Electrodynamics of Moving Matter

In this problem, we will explore how the motion of dielectric materials alters the electromagnetic fields inside of them. Work in natural units.

In materials, we write Maxwell's equations as

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \end{aligned}$$

You may assume these equations hold in all reference frames. Define the tensors:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

$$D^{\mu\nu} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & H_z & -H_y \\ D_y & -H_z & 0 & H_x \\ D_z & H_y & -H_x & 0 \end{pmatrix}.$$

(a) Argue that the equations of electrodynamics become

$$\begin{aligned} D^{\mu\nu}{}_{,\mu} &= J^\nu, \\ F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} &= 0. \end{aligned}$$

In its rest frame, a fluid[1] has permittivity ϵ and permeability μ. Suppose it is moving with a velocity u^μ in the lab frame.

(b) Show that

$$\begin{aligned} D^{\mu\nu} u_\nu &= \epsilon F^{\mu\nu} u_\nu, \\ F_{\mu\nu} u_\lambda + F_{\nu\lambda} u_\mu + F_{\lambda\mu} u_\nu &= \mu(D_{\mu\nu} u_\lambda + D_{\nu\lambda} u_\mu + D_{\lambda\mu} u_\nu). \end{aligned}$$

Now, assume the fluid flow is nonrelativistic, i.e. $u^\mu \approx (1, \mathbf{v})$.

(c) Show that the lowest order corrections to the constituent relations are

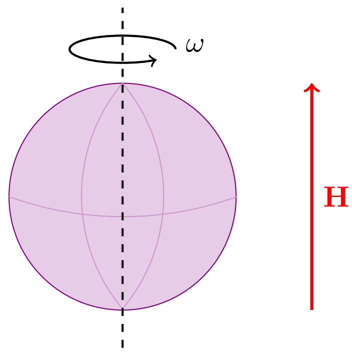
$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} + (\epsilon\mu - 1)\mathbf{v} \times \mathbf{H}, \\ \mathbf{B} &= \mu \mathbf{H} - (\epsilon\mu - 1)\mathbf{v} \times \mathbf{E}. \end{aligned}$$

[1] In relativity, the notion of a solid has no meaning, so all materials are fluids!

(d) Show that the boundary conditions between two media are given by

$$
\begin{aligned}
(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} &= 0, \\
(\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} &= 0, \\
(\mathbf{E}_2 - \mathbf{E}_1) \times \hat{\mathbf{n}} &= (\mathbf{v} \cdot \hat{\mathbf{n}})(\mathbf{B}_2 - \mathbf{B}_1), \\
(\mathbf{H}_2 - \mathbf{H}_1) \times \hat{\mathbf{n}} &= (\mathbf{v} \cdot \hat{\mathbf{n}})(\mathbf{D}_2 - \mathbf{D}_1).
\end{aligned}
$$

Now, let's use our newfound results to solve an interesting problem involving the nonrelativistic electrodynamics of a moving dielectric. A nonmagnetic rigid sphere of radius R and electric susceptibility χ rotates about an axis with angular frequency ω. It is in a uniform $\mathbf{H}$ field, oriented parallel to the rotation. There is no free charge in the sphere.



(e) Why can we write $\mathbf{E} = -\nabla\varphi(\mathbf{r})$ for a scalar potential?

(f) Show that

$$
\varphi(\mathbf{r}) = \begin{cases} \dfrac{2\chi\omega H}{3(5+2\chi)} r^2 \mathrm{P}_2(\cos\theta) + \dfrac{\chi\omega H}{3(1+\chi)}\left(R^2 - r^2\right) & r \le R \\ \dfrac{2\chi\omega H}{3(5+2\chi)} \dfrac{R^5}{r^3} \mathrm{P}_2(\cos\theta) & r > R \end{cases}.
$$