5D Chern-Simons Instability

In 5 spacetime dimensions, the action for electromagnetism can pick up a new contribution, called the *Chern-Simons term*:[1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{6}\epsilon^{\mu\nu\lambda\rho\sigma}A_\mu F_{\nu\lambda}F_{\rho\sigma}$$

Here α is the Chern-Simons coupling constant.

(a) Show explicitly that although $\mathcal{L}$ is not gauge invariant under $A_\mu \to A_\mu + \partial_\mu\chi$, the action $S = \int \mathrm{d}^5x\mathcal{L}$ is gauge invariant. Thus, we are allowed to add the Chern-Simons term to our action.

(b) Compute the correction to Maxwell's equations due to the Chern-Simons term.

(c) Suppose that we turn on a magnetic field: $F_{34} = B$. Linearize A and F around this solution and compute the dispersion relation for propagating waves. Show that the dispersion relation (written in terms of $\omega, k_1, k_2, k_3, k_4$) is

$$\omega^2 - k_1^2 - k_2^2 = \left(2|\alpha B| + \sqrt{k_3^2 + k_4^2 + 4\alpha^2 B^2}\right)^2.$$

In particular, discuss the limit where k_1, k_2 are small, and $k_3 = k_4 = 0$. What would be the effective equation of motion for $A_\mu(t, x_1, x_2)$ in this limit? Is there any tension between your answer and the requirement of gauge invariance of electromagnetism?

(d) Now, turn on an electric field: $F^{01} = E$. Show that the dispersion relation is

$$\omega^2 - k_1^2 = (k_\perp \pm 2\alpha E)^2 - 4\alpha^2 E^2$$

where $k_\perp \equiv \sqrt{k_2^2 + k_3^2 + k_4^2}$. Show that for $k < k_\mathrm{c}$, this solution is *unstable*.

It has been conjectured that this instability has consequences for the emergence of (possibly) spatially modulated phases in condensed matter physics, through some analogous between condensed matter physics and string theory-inspired constructions of quantum field theories in 4D space via 5D gravitational theories (where this Chern-Simons term can occur).

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[1] Curiously, it turns out that this Chern-Simons term is effectively *topological* – it can be written in a way that is manifestly independent of the curvature of the space. You do not have to worry about this fact for the purposes of this problem.