Laser Beam Breakup

At extremely high intensities, many materials have intensity-dependent indices of refraction. Consider a material which has a nonlinear polarization of the form

$$\mathbf{P} = \epsilon_0 \alpha |\mathbf{E}|^2 \mathbf{E}.$$

Thus as $\mathbf{E} \to \mathbf{0}$, the material acts like the vacuum.

In this problem, let us assume all waves are polarized in the x direction, and have amplitudes given by

$$E_0 = A_0(z) \mathrm{e}^{\mathrm{i}k(z-ct)} + \text{c.c},$$

with $|\mathrm{d}E_0/\mathrm{d}z| \ll |kE_0|$.

(a) Show that the wave equation approximately reduces to

$$\nabla_\perp^2 E + 2\mathrm{i}k \frac{\partial E}{\partial z} = -\alpha k^2 |E|^2 E.$$

Now, let us make the following ansatz:

$$E - E_0 = \mathrm{e}^{\mathrm{i}k(z-ct)} \left[A_+(z) \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} + A_-(z) \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \right] + \text{c.c.}$$

Here $\mathbf{q} \cdot \hat{\mathbf{z}} = 0$, and we assume that $A_+, A_- \ll A_0$.

(b) Solve for $A_0(z)$, neglecting the $A_\pm$ terms, and assuming the boundary condition $A_0(0) = a_0$.

(c) Now, linearizing the nonlinear wave equation, find a matrix equation for $\partial_z A_+$ and $\partial_z \overline{A}_-$.

(d) Show that the solutions to the matrix equation above are proportional to $\mathrm{e}^{\lambda z}$, and find an expression for λ.

(e) Show that when $a_0 > a_\mathrm{c}$, there is an instability. Sketch $\mathrm{Re}(\lambda)$ vs. q for these cases.

When this happens, typically the laser beam will fragment into many smaller laser beams, each of which have amplitude of about a_0.