Optical Invsibility

One can mathematically prove that nothing can remain invisible to waves solving the full wave equation. However, in 2006 it was shown that within the limit of geometric optics (which should be exponentially accurate at short wavelengths), *invisibility is possible* through a technique called optical conformal mapping, which we will outline in this problem through a simple example.

In this problem, assume that we have waves propagating in a 2 dimensional space (x, y). The index of refraction n is a function only of x and y. Let $z = x + \mathrm{i}y$, and suppose that

$$n = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|$$

for some analytic function $w(z)$.

(a) If c is the wave speed and ω is the wave frequency, and $k = \omega/c$, show that in complex coordinates defined by the analytic map $w(z)$, the wave equation reduces to

$$4\frac{\partial^2\psi}{\partial w \partial \overline{w}} + k^2\psi = 0.$$

We see that we have mapped the problem to flat space wave propagation. In flat space, the light "rays" follow straight lines, and so we just have to study the behavior of straight lines under the conformal mapping. Let us assume that $n \to 1$ as $|z| \to \infty$; this should mean that $w \approx z$ at large z. A theorem from complex analysis says that the only nontrivial choices of w occupy several Riemann sheets: i.e., there are branch cuts and each w could correspond to multiple z. We will exploit the multi-sheeted nature of this transformation to create devices which are invisible. Also, remember that so far everything we have done is *exact* for scalar waves – we will see the step where geometric optics must be invoked a bit later.

But first, let us consider a simple example of a nontrivial mapping w:

$$w = z + \frac{a^2}{z}.$$

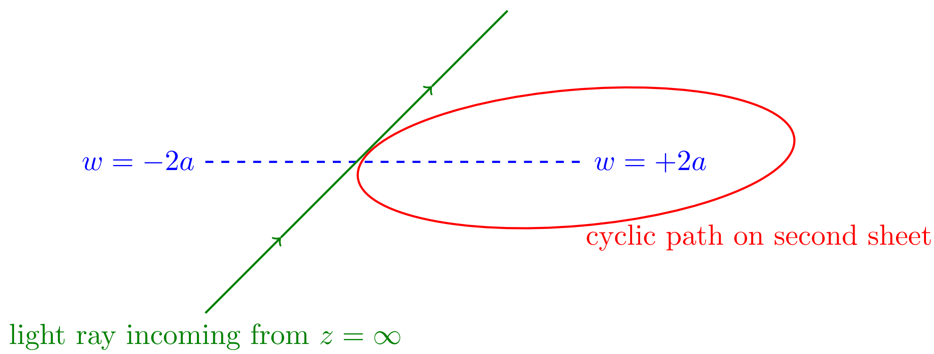
By solving a quadratic equation, it is clear that there are 2 choices of z for any given w – so we are dealing with a double cover of the complex plane.

(b) Show that one sheet of the w-plane maps to $|z| < a$, and one maps to $|z| > a$.

(c) Now, in the w plane, consider a light ray that passes over the branch cut between the points at $w = -2a$ and $w = +2a$. Argue by requiring consistency of ray propagation in the z plane that such a ray is a ray which truncates at $z = 0$, traveling in/out from $z = \infty$. Show that a ray which avoids the branch cut will never reach $|z| < a$, if we start the ray on the sheet containing $z = \infty$.

Now, here's where we invoke the trick of geometric optics: recall that the geometric optics approximation is that light "rays" are like particles traveling through an effective potential. We will also use the fact that we can choose $n_w(w)$, the index of refraction on the double cover, to be 1 on one sheet (the $z = \infty$ sheet), and a non-constant function on the other sheet. In particular, by using the approximation of geometric optics, we just need to pick $n_w(w)$ on the second sheet to be a function with periodic orbits for classical particles!

In particular, for the example we've been working with, let's engineer n_w on the second sheet so that it depends only on the distance from one of the branch points: say $w = +2a$. The goal is for orbits to take a periodic loop through the second sheet and exit the second sheet, back on to the first sheet, at exactly the same location, as shown in the figure below.



(d) What are the choices of n_w such that all "orbits" (light rays) have periodic paths? You will need to translate between the language of optics and quantum/classical mechanics.[1]

(e) Show that, even after accounting for refraction, ignoring the possibility of internal reflection, the outgoing light ray will exit in the w plane *exactly* as if it had never been refracted at all, given that the conditions of (d) are satisfied.

(f) Show that only for certain choices of the Kepler potential will it be impossible for any light, incoming from $z = \infty$, to totally internally reflect off the boundary between $n = 1$ and $n = n_w$, at the interface between the two Riemann sheets. Show that this is never possible for any harmonic oscillator choices of n_w. Conclude by describing the invisible region in the z plane, for appropriate Kepler choices of n_w.

More general conformal mappings to multi-sheeted Riemann surfaces could be used to mask other 2D shapes, but the general principles are the same.

[1] Use quantum mechanics as an analogy. Remember there will be overall coefficients that are unknown! There are *exactly* two potentials in classical mechanics for which all orbits are periodic around the origin of radial symmetry $w = +2a$: the harmonic oscillator and the Kepler $(1/r)$ potential.