Radiation Reaction of a Charged Sphere

One might ask whether or not bad behavior of the radiation reaction force is due to the point-like nature of the charges, just as there are infinite energies associated with sending the size of a charged object to zero. Let us consider a sphere of mechanical mass m_0, radius a with charge q uniformly distributed on its surface, moving at a *non relativistic* velocity $\mathbf{v}$, in 3 spatial dimensions.

(a) Show that for a *general* extended object with charge density $\rho(\mathbf{x})$ *in its rest frame*, whose origin is located at point $\mathbf{R}(t)$ in the spacetime, has a radiation reaction force:

$$\mathbf{F}_{\mathrm{rr}} = -\frac{q^2}{6\pi\epsilon_0 c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n n!}\gamma_n \frac{\mathrm{d}^{n+2}\mathbf{R}}{\mathrm{d}t^{n+2}}$$

where

$$\gamma_n \equiv \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{y}\rho(\mathbf{x})\rho(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{n-1}.$$

(b) Determine for γ_n for the uniformly charged sphere. Conclude that

$$\mathbf{F}_{\mathrm{rr}} = m_{\mathrm{E}}\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} + m_{\mathrm{E}}\frac{c}{2a}\left[\frac{\mathrm{d}\mathbf{R}(t-2a/c)}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right].$$

where

$$m_{\mathrm{E}} = \frac{q^2}{6\pi\epsilon_0 c^2 a}.$$

(c) Does the limit $a \to 0$ reduces to the famous formula we derived earlier?

Let us now see if there are badly behaved solutions to this "delay differential equation". To do this, we make the ansatz:

$$\mathbf{R}(t) = \mathbf{R}(0)\mathrm{e}^{\alpha t}$$

and ask what values of α satisfy the equation of motion. If there are any solutions with $\mathrm{Re}(\alpha) > 0$, these correspond to the runaway solutions.

(d) Show that there is a critical size a_{c}, so that when $a > a_{\mathrm{c}}$, there are no runaway solutions. Does anything interesting physically happen at this critical size?