Radio Waves in the Atmosphere

In many real world systems, we are interested in looking at the propagation of electromagnetic waves over regions of space where the index of refraction is highly variable. Consider an arbitrary set-up where $\mu = \mu_0$ and $\epsilon = \epsilon(\mathbf{r})$.

(a) Using Maxwell's equations, write down wave equations for **E** and **B**, including the effects of variable ϵ .

An important application of this is to the propagation of radio waves in the atmosphere. In the atmosphere, we approximately have $\epsilon = \epsilon_0 n(z)^2$. Furthermore, assume we are looking for propagating waves of the form $e^{i(kx-\omega t)}$ that are independent of y.

(b) Show that if $F_y = E_y$ and $F_z = n(z)E_z$,

$$\frac{\mathrm{d}^2 F_i}{\mathrm{d}z^2} + q_i(z)^2 F_i = 0$$

where

$$q_y^2 = \frac{\omega^2 n^2}{c^2} - k^2,$$

$$q_z^2 = \frac{\omega^2 n^2}{c^2} - k^2 + \frac{d^2 n}{dz^2} - \frac{2}{n^3} \left(\frac{dn}{dz}\right)^2.$$

A simple model of the Earth's atmosphere has, for z > 0,

$$n(z) = 1 + \frac{1}{2}\alpha z^2.$$

For the remainder of the problem, assume that $\alpha z^2 \ll 1$ in any regime of interest, so you can expand q_i to lowest non-constant order in z. Furthermore, assume that $dn/dz \ll q_i$. This allows you to adapt the WKB approximation technique of quantum mechanics to find approximate solutions.

- (c) Show that transverse waves (of E_y) will propagate infinitely far through the atmosphere, whereas for E_z for $\omega < \omega_c$ waves will be reflected by the atmosphere. Find an expression for ω_c .
- (d) For longitudinal waves with $\omega < \omega_c$, we define the effective height h_{eff} of the atmosphere to be the height such that $h_{\text{eff}} = cT/2$, where T is the time it takes for a ray of light to propagate from z = 0 (the ground) to its maximum height. Find an expression for h_{eff} in terms of ω , ω_c , k and c and sketch the result as a function of ω .