Stimulated Brillouin Scattering

Stimulated Brillouin scattering ccurs when fluctuations, such as phonons, cause scattering of an incident light beam and the phonons themselves are excited by the incident light. In this problem, we'll work through a calculation showing how a laser beam, e.g., can have its intensity amplified as it passes through a material, under special conditions.

Electrostriction refers to the compression of a material when an electric field is applied. Suppose a material has a permittivity ϵ which is dependent on its mass density ρ.

(a) Suppose there is a constant electric field $\mathbf{E}$. Using the fact that $\mathrm{d}W = P\mathrm{d}V$ and $u = \frac{1}{2}\epsilon(\rho)\mathbf{E}^2$, show that the electrostrictive pressure is

$$P_{\mathrm{es}} = -\frac{1}{2}\gamma\mathbf{E}^2$$

and find an expression for γ.

(b) The compressibility is defined by

$$\kappa \equiv \frac{1}{\rho}\frac{\partial\rho}{\partial P}.$$

Show that after the electric field is applied, the change in density from initial density ρ_0 is given by

$$\Delta\rho = \frac{1}{2}\rho_0\gamma\kappa\mathbf{E}^2.$$

(c) Suppose we apply a time-dependent field of the form

$$\mathbf{E} = \mathbf{E}(\omega)\mathrm{e}^{-\mathrm{i}\omega t} + \overline{\mathbf{E}(\omega)}\mathrm{e}^{\mathrm{i}\omega t}.$$

Show that the time-averaged effective polarization induced is

$$\mathbf{P} = \frac{\epsilon_0}{\epsilon}\kappa\gamma^2|\mathbf{E}(\omega)|^2\mathbf{E}.$$

Suppose we now have a incident beam, $\mathbf{E}_1 = E_1\mathrm{e}^{\mathrm{i}(k_1x-\omega_1t)}\hat{\mathbf{n}}$ propagating in the medium, along with a reflected beam $\mathbf{E}_2 = E_2\mathrm{e}^{\mathrm{i}(-k_2x-\omega_2t)}\hat{\mathbf{n}}$, scattered off of a phonon traveling rightwards with frequency Ω. Assume the phonon obeys a dispersion relation $\Omega = qv$ (here, $\mathbf{q}$ is the wave vector of the phonon), and the light obeys a dispersion relation $\omega = ck/n$, where $n^2 = \epsilon/\epsilon_0$.

(d) Explain why it makes sense that $\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{q}$, and $\Omega = \omega_1 - \omega_2$.

(e) Assuming that $v \ll c/n$, show that $\mathbf{k}_1 \approx -\mathbf{k}_2 \approx \mathbf{q}/2$.

(f) Suppose that the material density obeys the PDE

$$\frac{\partial^2\rho}{\partial t^2} - v^2\nabla^2\rho - \Gamma\frac{\partial}{\partial t}\nabla^2\rho = \nabla^2 P_{\mathrm{es}}.$$

Show that, if we look only at the fluctuations of the form $\alpha\mathrm{e}^{\mathrm{i}(qx-\Omega t)} = \rho$ for constant α, we get the approximate result

$$\alpha = \frac{q^2}{{\Omega_{\mathrm{B}}}^2 - \Omega^2 - \mathrm{i}\omega q^2\Gamma}\frac{\epsilon_0}{\epsilon}\gamma E_1\overline{E_2}$$

Here we have defined $\Omega_{\mathrm{B}} \equiv 2nv\omega_1/c$.

Now we combine the results of the previous parts to find an expression for how the intensity of an applied electric field grows as it passes through the material. We have the general wave equation

$$\frac{\partial^2 E}{\partial x^2} - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{\epsilon_0 c^2} P.$$

Here P is the polarization, not the pressure!

(g) By re-considering the argument used in part (c), show that if E_1 and E_2 depend slowly on z (compared to the wavelength), the time-independent solution to the wave equation has the form

$$\frac{\mathrm{d}E_1}{\mathrm{d}x} = \frac{\mathrm{i}\omega\gamma}{2nc\epsilon\rho_0}\alpha E_2,$$
$$\frac{\mathrm{d}E_2}{\mathrm{d}x} = -\frac{\mathrm{i}\omega\gamma}{2nc\epsilon\rho_0}\overline{\alpha} E_1.$$

(h) Defining intensities $I_i = 2n\epsilon_0 c|E_i|^2$, show that

$$\frac{\mathrm{d}I_1}{\mathrm{d}x} = \frac{\mathrm{d}I_2}{\mathrm{d}x} = -gI_1 I_2$$

and find the expression for the constant g.

(i) Verify that $I_1 - I_2$ is z-independent. Set this constant equal to C.

(j) Given C and $I_1(0)$, find $I_1(z)$ by solving the above ODEs.

(k) An SBS amplifier helps to (ironically?) reduce the intensity of the incident wave significantly. An amplifier of length L works as follows: in addition to the input at $x = 0$, add a small wave at ω_2 traveling left at $x = L$ with intensity ηI_0. Use the equation from the previous part to find a transcendental expression relating $I_1(L)$ to η, g, L, and I_0.

(l) Use a computer program to plot $1 - I_1(L)/I_0$ vs. gLI_0, for $\eta = 10^{-2}$ and 10^{-6}. Comment on the results.