Stimulated Raman Scattering

Stimulated *Raman scattering* occurs when a beam of light passing through a material interacts with the molecules of that material, helping to drive transitions between the various electronic states in those molecules. This problem will show you how Raman scattering works in a classical picture.

Let the number of molecules per unit volume be N. The molecules inside of the material can be approximated as dipoles, with the strength of the dipole moment related to the displacement of the molecules from their equilibrium position, x. For simplicity, assume x obeys a harmonic oscillator equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + {\omega_\mathrm{a}}^2 x = \frac{F}{m},$$

and that the polarization of the medium is given by

$$P = \epsilon_0 N\eta x E.$$

Suppose we shine light on this medium which, locally, has the form

$$\mathbf{E}(t) = (E_0 + E'\mathrm{e}^{\mathrm{i}\delta t})\mathrm{e}^{-\mathrm{i}\omega t}\hat{\mathbf{n}} + \text{complex conjugate.}$$

(the direction of the light's polarization will be ignored for this problem).

(a) Before getting into the specifics of this model, let's derive a generic fact. Suppose we have some nonlinear polarization $\mathbf{P}_\mathrm{NL}$ in our model, such that at a given frequency and wave number: $\mathbf{D}(k,\omega) = \mathbf{E}(k,\omega) + \epsilon_0\mathbf{P}_\mathrm{NL}(k,\omega)$. Show that, for a wave propagating in the z direction, of the form

$$\mathbf{E} = \mathbf{E}_0(z)\mathrm{e}^{\mathrm{i}\omega(z/c-t)},$$

then Maxwell's equations imply, if $\mathbf{P}_\mathrm{NL}$ is small,

$$\frac{\mathrm{d}\mathbf{E}_0}{\mathrm{d}z} = \frac{\mathrm{i}\omega}{2c\epsilon_0}\mathbf{P}_\mathrm{NL}.$$

(b) Now, let's turn back to our model, and neglect the vector indices on polarization and electric field from now on. The dipole moment of a single molecule is $p = \epsilon_0 N\eta xE$. Using this result, compute the average energy, averaging over the ω oscillations *but not* the δ oscillations, of this single molecule, and from that compute the force term $F(t)$.

(c) Given $F(t)$, find $x(t)$.

(d) Find an expression for $\mathbf{P}$ using the x found in part (b). The incident light contains frequencies of $\pm\omega$ and $\pm(\omega+\delta)$; what frequencies does the outgoing light contain?

(e) Show that when $\delta = \omega_\mathrm{a} > 0$, the solution to the equation of part (a) for the $+(\omega-\omega_\mathrm{a})$ frequency component of the electric field is well-approximated by $\mathbf{E} = E'(x)\mathrm{e}^{\mathrm{i}\omega x/c}\hat{\mathbf{n}}$, where

$$\frac{\mathrm{d}E'}{\mathrm{d}z} = K|E_0|^2E'$$

and $K > 0$ is a real constant. Find the expression for K.

Under the circumstances of part (e), the light beam has "pure gain", i.e. the intensity of the lower frequency component of the incident beam increases without phase shifting as it passes through the material.