electromagnetism → electromagnetic waves ★★★

Transfer Matrices in Optics

In a real world analysis of a complex optical system, one of the most useful tricks is the introduction of a transfer matrix. In this problem, we'll see how to construct and use transfer matrices for normally incident electromagnetic waves in a large optical system.

Suppose we have an optical system which is translationally invariant in the y and z directions. We can write the electric field of a normally incident plane wave therefore

$$E = E_+ \mathrm{e}^{\mathrm{i}n\omega x/c} + E_- \mathrm{e}^{-\mathrm{i}n\omega x/c}.$$

The polarization of the wave does not concern us. Suppose that at some initial state, therefore, our electric field is described by the vector (E_{+1}, E_{-1}), and at some later time by the vector (E_{+2}, E_{-2}). (The vector refers to whether the wave is moving forwards/backwards, not the vector nature of $\mathbf{E}$). If our system is linear, we then construct a 2×2 matrix T such that

$$\begin{pmatrix} E_{+2} \\ E_{-2} \end{pmatrix} = T \begin{pmatrix} E_{+1} \\ E_{-1} \end{pmatrix}$$

T is called a **transfer matrix**. To start off, let's calculate some sample transfer matrices.

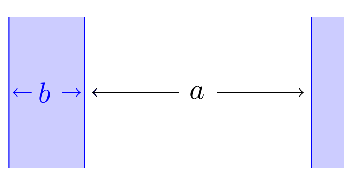
(a) Let $P(n; x)$ be the matrix representing propagation through a medium of index of refraction n for a distance x. Show that[1]

$$P(n; x) = \cos \frac{n\omega x}{c} + \mathrm{i} \sin \frac{n\omega x}{c} \sigma_3.$$

(b) Let $B(n_1 \to n_2)$ be the matrix representing a wave propagating from a medium of index n_1 ($x < x_0$) to a medium with index n_2 ($x > x_0$). Show that

$$B(n_1 \to n_2) = \frac{1 + n_1/n_2}{2} + \frac{1 - n_1/n_2}{2} \sigma_1.$$

Now suppose we are considering a stack of N thin films of index of refraction n, each with thickness b. The films are separated by air ($n = 1$) of thickness a. The transfer matrix formalism is ideal to analyze this problem in a nearly-exact manner.



(c) Let $\alpha = \omega a/c$ and $\beta = n\omega b/c$; let U be the transfer matrix for one unit of the thin film stack (as shown in the figure). Show that

$$\begin{aligned} U = \frac{1}{4n} \Big[&\left((n+1)^2 \cos(\alpha + \beta) - (n-1)^2 \cos(\alpha - \beta) \right) \\ &+ 2(n^2 - 1) \sin\beta \sin\alpha \sigma_1 - 2(n^2 - 1) \sin\beta \cos\alpha \sigma_2 \\ &+ \left((n+1)^2 \sin(\alpha + \beta) - (n-1)^2 \sin(\alpha - \beta) \right) \mathrm{i}\sigma_3 \Big]. \end{aligned}$$

[1] σ_i represents a Pauli spin matrix in this problem (think back to quantum mechanics for their properties).

(d) If we build a stack such that $\alpha = \beta = \pi/2$, compute T and show that

$$\frac{I_{\text{trans}}}{I_{\text{inc}}} = \frac{4}{\left(1 + \dfrac{1}{n^2}\right)^N}.$$

Comment on this result, and on its possible usefulness in an application.