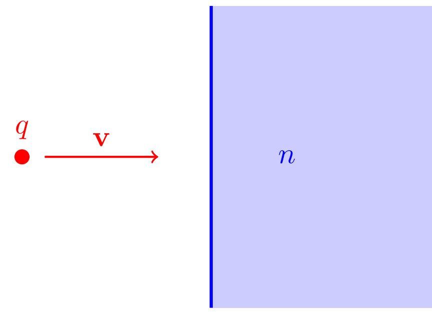
Transition Radiation

A charge q moves at constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$ from a vacuum ($x < 0$) into a linear dielectric ($x > 0$) with index of refraction $n = \sqrt{\epsilon/\epsilon_0}$, which you can assume is real. Assume that $v < c/n$.



(a) Assume that at time $t = 0$, the charge is at $x = y = z = 0$. Fourier transform Maxwell's equations in the y, z and t directions, and show that the electric field $\mathbf{E}_\mathrm{s}$ generated by the moving source, ignoring the presence of the boundary, is given by

$$\mathbf{E}_\mathrm{s}(x, k_y, k_z, \omega) = \frac{\mathrm{i}q}{\epsilon_0 n^2 v} \frac{\omega v \left(\dfrac{n^2}{c^2} - \dfrac{1}{v^2}\right)\hat{\mathbf{x}} - \mathbf{k}}{k^2 - \omega^2 \left(\dfrac{n^2}{c^2} - \dfrac{1}{\omega^2}\right)} \mathrm{e}^{\mathrm{i}\omega x/v}$$

In this equation, take n^2 to be a function of x.

(b) As you may have noticed, the above does not quite solve Maxwell's equations, because of the boundary at $x = 0$, where the abrupt change in n will cause $\mathbf{E}_\mathrm{s}$ to be discontinuous. We can remedy this by writing the true electric field as $\mathbf{E} = \mathbf{E}_\mathrm{s} + \mathbf{E}_\mathrm{r}$, where $\mathbf{E}_\mathrm{r}$ is a radiation field, added to satisfy the boundary conditions. Show, on general principles, that the form of $\mathbf{E}_\mathrm{r}$ must be

$$\mathbf{E}_\mathrm{r} = (x, k_y, k_z, \omega) = \mathrm{i}A_\pm(\mathbf{k}, \omega)\left[\hat{\mathbf{x}} \mp \frac{\mathbf{k}}{k^2}\sqrt{\frac{n^2\omega^2}{c^2} - k^2}\right] \mathrm{e}^{\pm\mathrm{i}x\sqrt{n^2\omega^2/c^2 - k^2}}$$

where the top sign is used for $x > 0$, and the bottom sign for $x < 0$.

(c) The boundary condition at $x = 0$ provides equations which can be solved to find $A_\pm$. Write down these equations, but do not solve them.

If you did the nasty algebra, you would find

$$A_- = \frac{q\beta\kappa^2(n^2 - 1)(1 - \beta^2 + \beta\sqrt{n^2 - \kappa^2})}{\omega(1 - \beta^2 + \beta^2\kappa^2)(1 + \beta\sqrt{n^2 - \kappa^2})(\sqrt{n^2 - \kappa^2} + n^2\sqrt{1 - \kappa^2})}$$

where $\beta \equiv v/c$ and $\kappa \equiv kc/\omega$.

For the remainder of the problem, our goal will be to compute the energy W radiated by this particle as it moves across the boundary, in the limit of $t \to \infty$. This is the so called **transition radiation**. To do this, we begin with the formula (note that the factor of 2 comes from the contribution of the magnetic field):

$$W = \int_{x<0} \mathrm{d}V \; \epsilon_0 \mathbf{E}_\mathrm{r} \cdot \overline{\mathbf{E}_\mathrm{r}}.$$

This is basically because in the large time limit, only the radiation field is present at $x < 0$. You do not need to justify this statement, although it is not difficult.

(d) Begin by showing that

$$W = \int \mathrm{d}^2\mathbf{k}\mathrm{d}\omega \; \frac{\epsilon_0}{(2\pi)^4} \frac{\omega^2}{ck^2} \sqrt{1 - \frac{c^2k^2}{\omega^2}} |A_-|^2.$$

(e) Now justify, the following statement: the only relevant integral is over momenta such that $ck \le \omega$. Switching variables to

$$\sin\theta \equiv \frac{ck}{\omega},$$

re-write the integral for W as an integral only over θ and ω.

(f) Work in the limit where $n \approx 1$ and $v \approx c$. Show that, in this limit, we can approximate

$$W \sim \left(\frac{n-1}{n+1}\right)^2 \log \frac{1}{1 - v^2/c^2}.$$

Comment on the result. How much of the particle's energy (relatively) is lost to transition radiation as it crosses the boundary, as the particle becomes more energetic?