Waveguide with Perturbed Shape

Consider the eigenvalue problem for the Laplacian in a 2-D region R where we have an exact solution (for specified boundary conditions):

$$\nabla^2 f_\alpha = -{\lambda_\alpha}^2 f_\alpha.$$

As always, f_α is orthogonal to f_β if $\alpha \neq \beta$. Now suppose we slightly perturb the *boundary* of our region, so that we have some new region R'. The exact solution of the eigenvalue problem in R' (for the same boundary conditions) is

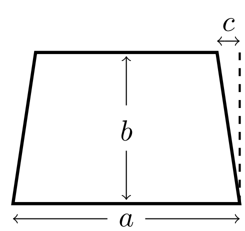
$$\nabla^2 f'_\alpha = -{\lambda'_\alpha}^2 f_\alpha.$$

We approximate that $f'_\alpha \approx f_\alpha + \delta f_\alpha$, where $\delta f_\alpha \ll f_\alpha$. Furthermore, let δs represent the shortest distance from ∂R to $\partial R'$, at each point on ∂R. Assume that δs is small enough that f_α can be accurately approximated by a Taylor series on that length scale.

(a) Show that

$${\lambda'_\alpha}^2 \approx {\lambda_\alpha}^2 + \frac{\displaystyle\oint_{\partial R} \left[f_\alpha \frac{\partial^2 f_\alpha}{\partial n^2} - \left(\frac{\partial f_\alpha}{\partial n} \right)^2 \right] \delta s \, \mathrm{d}l}{\displaystyle\int_R {f_\alpha}^2 \, \mathrm{d}S}.$$

(b) Now consider the following waveguide with perfectly conducting walls, where $a, b \gg c$:



Find the lowest order correction to the fundamental frequencies of all TE and TM modes. Does this perturbation break the degeneracy between TE and TM modes (of the same m, n)?[1]

[1] Technically, we should first say that we want a/b to be irrational, so that there will never be degeneracy between two TE modes or two TM modes (in the unperturbed problem). Such degeneracy complicates the picture.