Gaussian Beams

The basic theory of electromagnetic radiation suggests that if a point source creates electromagnetic radiation, it very rapidly attenuates: the intensity of the beam falls off as r^{-2}. This is certainly true, so how do we make a laser beam, which can appear so focused?

The answer is to use a Gaussian beam. This problem will guide you through understanding how Gaussian beams are proper solutions to the wave equation (in a special limit) with very interesting properties. For simplicity, we may consider the geometrc optic limit, and consider a scalar wave $\psi(x, y, z, t)$ propagating as

$$\psi(x, y, z, t) = A(x, y, z)\mathrm{e}^{\mathrm{i}k(z-ct)}.$$

Suppose that

$$k \gg \frac{\partial A}{\partial z}.$$

(a) Show that approximately,

$$0 = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + 2\mathrm{i}k\frac{\partial A}{\partial z}.$$

This is called the paraxial Helmholtz equation.

(b) Let $r^2 = x^2 + y^2$. Verify that the paraxial Helmholtz equation is satisfied by

$$A = \frac{C}{q(z)}\mathrm{e}^{\mathrm{i}kr^2/2q(z)}$$

for any constant C and function $q(z) = z + \alpha$, for any complex constant α.

(c) Show that writing $q(z)$ in the form

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{\mathrm{i}}{2kW(z)^2}$$

where $R(z)$ and $W(z)$ are real functions results in

$$A = A_0\frac{W_0}{W(z)}\mathrm{e}^{-r^2/4W(z)^2}\mathrm{e}^{\mathrm{i}kr^2/2R}\mathrm{e}^{\mathrm{i}\zeta(z)}$$

where $\zeta(z)$ is a specific function, and A_0 and W_0 are constants. The name Gaussian beam should now be pretty clear!

(d) Show that $W(z)$ has one minimum: we may use translation symmetry to fix this minimum at $z = 0$. Then show that

$$W(z) = W_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

and find an expression for W_0 in terms of k and z_0. Explain why z_0 is a free parameter of the problem.

(e) For $z \gg z_0$, explain why the Gaussian beam looks like a beam propagating outwards at some angle θ with respect to the z-axis, and find an expression for θ. Explain how we can make θ as small as we want. This corresponds to making a laser beam which stays very focused.