Green's Reciprocity Principle

Suppose we have a region of space R, and that if we place charge density ρ_1 in R and surface charge density σ_1 on ∂R, we obtain φ_1 by solving Poisson's equation; similarly, we get φ_2 for charge densities ρ_2 and σ_2 in R and ∂R repsectively.

Green's reciprocity theorem states that

$$\int_R \rho_1\varphi_2 \mathrm{d}V + \int_{\partial R} \sigma_1\varphi_2 \mathrm{d}S = \int_R \rho_2\varphi_1 \mathrm{d}V + \int_{\partial R} \sigma_2\varphi_1 \mathrm{d}S.$$

(a) Prove Green's reciprocity theorem from first principles.

Green's reciprocity theorem has some clever uses. For example, use it to solve the following problem:

(b) A charge q is placed inside two perfectly conducting parallel planes, separated by a distance d. The charge is a distance a away from one of the plates. Find the charge induced on both plates.