Oblate Spheroidal Coordinates

Sometimes the introduction of an "exotic" set of coordinates allows us to solve problems that are outrageously difficult by other means. In this problem, we will introduce **oblate spheroidal coordinates** and use it to solve a strange electrostatics problem.

We define oblate spheriodal coordinates (ζ, ξ, ϕ) in terms of cylindrical coordinates; ϕ is the usual azimuthal angle, and

$$\begin{aligned} s &= a\sqrt{(1+\zeta^2)\,(1-\xi^2)}, \\ z &= a\zeta\xi. \end{aligned}$$

We restrict $\zeta \in [0, \infty)$ and $\xi \in [-1, 1]$. a is a parameter we can choose.

(a) Determine what each coordinate represents in this coordinate system.

The element of length in this coordinate system is

$$\mathrm{d}s = \frac{1}{a}\sqrt{\frac{1+\zeta^2}{\zeta^2+\xi^2}}\mathrm{d}\zeta + \frac{1}{a}\sqrt{\frac{1-\xi^2}{\zeta^2+\xi^2}}\mathrm{d}\xi + \frac{1}{a\sqrt{(1+\zeta^2)(1-\xi^2)}}\mathrm{d}\phi.$$

You can show, with a lot of work, that the Laplacian in this coordinate system acts on a scalar, such as the scalar potential, as

$$\begin{aligned} \nabla^2\varphi = {} & \frac{1}{a^2(\zeta^2+\xi^2)}\left[\frac{\partial}{\partial\zeta}\left((1+\zeta^2)\frac{\partial\varphi}{\partial\zeta}\right) + \frac{\partial}{\partial\xi}\left((1+\xi^2)\frac{\partial\varphi}{\partial\xi}\right)\right] \\ & + \frac{1}{a^2(1+\zeta^2)(1-\xi^2)}\frac{\partial^2\varphi}{\partial\phi^2}. \end{aligned}$$

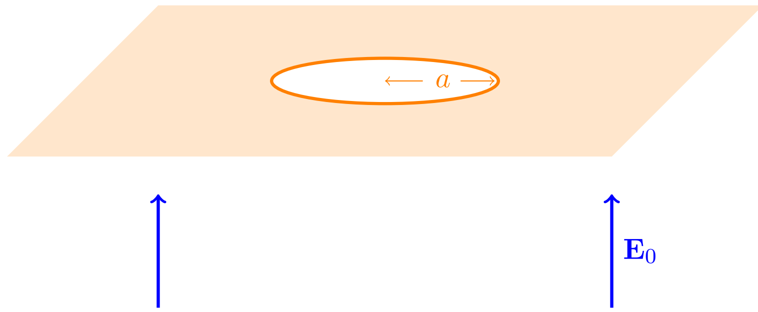
(b) Show that a solution to Laplace's equation independent of ϕ can be written as

$$\varphi = \sum_{n=0}^{\infty}\left(C_1\mathrm{P}_n(\xi) + C_2\mathrm{Q}_n(\xi)\right)\left(C_3\mathrm{P}_n(\mathrm{i}\zeta) + C_4\mathrm{Q}_n(\mathrm{i}\zeta)\right).$$

Here C_1, C_2, C_3 and C_4 are (possibly complex) constants and $\mathrm{P}_n(x)$ and $\mathrm{Q}_n(x)$ are the two linearly independent solutions to Legendre's equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + n(n+1)y = 0.$$

We can use these coordinates to solve the following electrostatics problem, which is *very* hard by other means: an infinitesimally thin conducting rectangular slab has a circular hole of radius a drilled out of it. Far below the hole, the electric field is normal to the plate of magnitude E_0; far above, there is no electric field.



(c) Show that the electric potential is given by[1]

$$\varphi = \frac{aE_0}{\pi}|\xi|\,(1+\zeta\arctan\zeta) - \frac{1}{2}aE_0\zeta\xi.$$

(d) Show that the total charge induced on the upper surface of the plate is

$$q_{\text{upper}} = -\frac{\pi}{2}a^2E_0.$$

[1]The following functions may be useful: $\mathrm{P}_0(x)=1$, $\mathrm{P}_1(x)=x$, $\mathrm{Q}_0(x)=\log\sqrt{(1+x)/(1-x)}$, $\mathrm{Q}_1(x)=x\mathrm{Q}_0(x)-1$. Be careful with the boundary conditions at $z=0$!