Thickness of the Cell Membrane

In this problem, we will explore a classic experiment used to measure the thickness of the cell membrane, before electron microscopy allowed for imaging the structure at the atomic/molecular level. Historically, the cell membrane was approximately modeled as a thin strip around the cell, filled with a liquid of permittivity $\epsilon \sim 3\epsilon_0$. The coefficient was not known precisely, but if we're only interested in an order of magnitude estimate, this is not a problem. A natural way to measure the thickness d of the cell membrane is to compute the capacitance per unit area c, which is a *geometric quantity*:

$$c = \frac{\epsilon}{d}.$$

To answer this problem (qualitatively), we model each cell as a sphere of radius R – the interior of the cell has radius $R-d$, with the final d corresponding to the cell membrane. A priori, you may assume that $R \gg d$ – of course, this couldn't have been known beforehand, and the problem can be solved without this approximation, but this will make some of the calculations simpler. We also assume that the inner part of the cell is an equipotential, as the membrane should shield the cell from external electric fields; we then assume that no net charge flows in or out of the outer wall of the cell membrane.

(a) Let us begin by imagining that we place this cell in an (asymptotic) constant external electric field of strength E_0. As there are no external charge densities, we simply must solve Laplace's equation to determine the electric potential φ everywhere. What are the boundary conditions on φ?[1] Show that, in spherical coordinates, the potential is given by

$$\varphi = \begin{cases} -E_0 r\cos\theta - E_0 R^3\cos\theta/2r^2 & r > R \\ -3E_0 R^3(r-(R-d)^3 r^{-2})\cos\theta/2(R^3-(R-d)^3) & R-d < r < R \\ 0 & R-d > r \end{cases}.$$

Conclude that the electric field is approximately radial inside of the cell membrane.

(b) Conclude that the potential energy density stored in the cell membrane is given by (approximately)

$$U = \frac{3\pi}{2} c R^4 E_0^2.$$

Given this prelude, let us now describe how we can calculate c. Imagine that we have a suspension of N cells inside of a tube of length L. Across this tube, we apply a (very slowly varying) time-dependent voltage

$$\varphi_{\text{ext}} = \varphi_0 \cos(\omega t).$$

We measure a current $I_0\cos(\omega t - \phi)$. The phase shift ϕ implies that the cells behave as capacitors.

(c) Explain why the stored energy in the cell membrane capacitors is given by

$$U = \frac{\varphi_0 I_0 \sin\phi}{2\omega}\cos^2(\omega t).$$

[1] Use Ohm's Law, if necessary, to determine the boundary condition at $r = R$.

(d) Show that

$$c = \frac{I_0 \sin\phi}{2\varphi_0 \omega} \frac{2L^2}{3\pi N R^4}.$$

The first fraction corresponds to macroscopic quantities which are easy to measure. We then need to know N and R to determine the capacitance. It was known that $R \sim 1$ μm, and working under the assumption that most of the tube is made up of cells, it is easy to determine N as well.

Experimentalists first measured the value $d \approx 3$ nm. The value is actually closer to about 9 nm, as it turns out that the assumption of a homogeneous ϵ is not perfect.