Edge Effects in a Capacitor

The goal of this problem is to explore, in a crude sense, the correction to the capacitance C of a parallel-plate capacitor due to edge effects. Without edge effects, we know that if the area of the plates is A and the distance between them is d, then

$$\frac{C}{\epsilon_0} = \frac{A}{d}.$$

The factor on the right hand side is a geometric factor and should not depend on any electrical quantities.

If $d \ll \sqrt{A}$, then on the length scale of d, near the edge of the capacitor, the capacitor looks like parallel semi-infinite plates separated by a distance d. Let us suppose that the upper plate is held at potential $\varphi_0/2$, and is located at a distance $y = d/2$ above the x-axis; the lower plate is held at potential $-\varphi_0/2$ and is located at $y = -d/2$. Suppose that $\delta\epsilon_0$ is the change in capacitance per unit length associated with such a scenario.

(a) Explain why we can approximate

$$\frac{C}{\epsilon_0} = \frac{A}{d} + L\delta$$

where L is the perimeter of the capacitor plate.

Now, we are in a position to actually compute the edge effects. To do so, we will utilize the trick of conformal mapping. Consider the conformal transformation

$$z = \frac{d}{\phi_0}w + \frac{d}{2\pi}\mathrm{e}^{2\pi w/\varphi_0}.$$

Here z represents the spatial coordinate, and w the complex electric potential.

(b) Describe this conformal mapping in detail. What range of the w complex plane is mapped to what (use the requirement that the mapping is invertible). Explain why $\varphi = \mathrm{Im}(w)$ is the solution to Laplace's equation we are looking for, given the boundary conditions specified earlier.

(c) Numerically plot the equipotential lines.

(d) Describe how in principle one can exactly calculate δ. You do not need to do this – instead, settle for a qualitative answer. Explain why

$$\delta = \frac{1}{2\pi}\log\frac{\sqrt{A}}{d} + \delta_0$$

where δ_0 is an unknown constant (independent of d or A). Note whether δ is positive or negative, and give a physical explanation for the answer.