Electrostatics in a Compactified Space

Consider electrodynamics in a 4 dimensional space. You can still assume that Gauss' Law holds, and that the electric field $\mathbf{E} = -\nabla\varphi$ can be written as the gradient of a potential.

(a) Consider a point charge of strength q_4 at the origin. Determine the electric potential $\varphi_4(r_4)$, where r_4 is the distance in 4D space from the point charge.[1]

In string theory, one of the fundamental ideas is that there may be extra dimensions to our universe. However, these extra dimensions are *compactified*, which means that the extra dimensions are only noticeable on very short length scales. Let us give a quantitative example of compactification. Let us denote the 4 directions of our 4D space by (x, y, z, w). To compactify the fourth dimension w, we demand that our electrostatic solution φ is periodic in w: $\varphi(w) = \varphi(w + 2\pi a)$. The physical interpretation of this is that the fourth dimension is in fact rolled up, or compactified, into a circle of radius a.

(b) Compactify the 4D space as described above onto a circle of radius a. Determine the new potential $\varphi_3(x, y, z) \equiv \varphi_4(x, y, z, 0)$, using the φ_4 you found earlier, but requiring consistency in compactification.[2]

(c) Show that when $r_3 \gg a$,

$$\varphi_3(r_3) \approx \frac{q_3}{4\pi r_3},$$

and relate q_3, the effective strength of the point charge in 3D space, to q_4. Comment on this result.

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[1] To do this, it is easiest to use the integral form of Gauss' Law, which holds in an arbitrary number of dimensions. Also note that for a potential $\varphi(r)$ in any number of dimensions, the electric field is purely radial and has magnitude given by $-\partial\varphi/\partial r$. Note that the surface area of a 3-sphere of radius R is $2\pi R^2$.

[2] Think about placing image charges so that the solution is consistent on the compactified space: i.e., is periodic in w with periodic $2\pi a$. To determine the exact form of φ_3, the following sum is helpful: $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{1 + (n\pi x)^2} = \frac{1}{x}\coth\frac{1}{x}$.