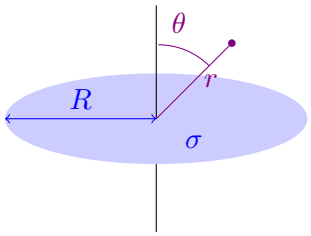
Electrostatics of a Disk

Consider a uniformly charged disk of radius R with surface charge density σ.



We would like to compute the electric potential $\Phi(r, \theta)$ as a function of the distance r from the center of the disk, and θ, the angle from the vertical axis. Normally such problems do not have exact solutions, although miraculously this one *does* through a change of coordinates to a bizarre coordinate system.

This is quite challenging, so instead we will be happy to just explore the asymptotic behavior of Φ, at generic θ. First, however, let's see what happens when $\theta = 0$.

(a) Show that

$$\Phi(r, \theta = 0) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2 + r^2} - r\right]$$

(b) Check that your answer reduces to what you would expect when $r \ll R$ and $r \gg R$.

Now, let us explore the $r \gg R$ limit for a generic value of θ. Later on, we will learn about the multipole expansion, which allows us to do an efficient asymptotic expansion of Φ for a problem such as this, and we will find that based on the symmetries of this problem, we expect that

$$\Phi = f_0(r) + f_2(r)\left(3\cos^2\theta - 1\right) + \cdots$$

where the $\cdots$ terms are subleading terms at large r, compared to the first two.

(c) Write down an explicit form for $\Phi(r, \theta)$ in integral form. Perform an asymptotic approximation to the integral where you keep terms only up to the leading order θ-dependent term (but make sure to retain all relevant terms at this order in r). Verify that you do indeed find the multipole form suggested above: what are $f_0(r)$ and $f_2(r)$?

(d) Focus now on only the *leading* correction at large r. Do you think you could have guessed this without going through this calculation?

Now, let's turn to a different problem. Suppose that we have an infinite plane with constant surface charge density σ, except for a circular hole of radius R drilled into the plane.

(e) How can we use the calculation we just did to "solve" this problem quickly? In particular, provide a good *sketch* of the electric fields in all relevant regions of space. Be sure to include what is happening both near and far from the hole.