Dirac Edge Modes in a Topological Insulator

In a theory of insulators with time reversal invariance and parity invariance, one can show that there is a classification of insulators into a trivial and non-trivial topological class. Exciting physics can happen at the surface between a normal and a topological insulator. In this problem, we will argue that at the edge between a topological and normal insulator in 2 spatial dimensions, there must exist a massless Dirac fermion living in effectively 1 spatial dimensions, that is localized at the boundary between the two insulators.

To begin, we outline some general arguments. The combination of parity and time reversal symmetry can be shown on general principles to lead to, on general symmetry arguments, an effective low energy theory of massive Dirac fermions, both in the non-trivial and trivial insulating phases. However, the sign of the mass is different in the non-trivial (positive) and trivial (negative) phases. In particular, if we place a non-trivial insulator in the half-plane $x > 0$, and a trivial insulator in the half-plane $x < 0$, the Hamiltonian for our theory becomes

$$H = m\gamma^t \mathrm{sign}(x) + \mathrm{i}\gamma^t\gamma^j\partial_j.$$

(a) Explain why we can essentially use the free particle solutions to the Dirac equation, up to gluing them together at the boundary. Pick a choice of γ matrices and write down the energy spectrum and eigenvectors.

(b) Show that there are usual Dirac fermion modes, with the exact same spectrum as in free space, and find the associated eigenvectors.

(c) Show that in addition, there is an *edge* mode, localized around $x = 0$, with the dispersion relation

$$E = \pm|k_x|.$$

This edge mode is a consequence of the "topological" nature of the two insulators; only when the mass changes sign across the interface does this edge mode exist. This mode is a free Dirac fermion; this remains true in higher dimensions.[1]

[1] Formally, one can show that even if there is a smooth change in $m(x)$, the massless Dirac fermion still exists.