Dirac Oscillator

In this problem, we will study the "Dirac oscillator" Hamiltonian in 2+1 dimensions. The Dirac oscillator equation of motion is given by the following modified Dirac equation:

$$\mathrm{i}\hbar\gamma^t\partial_t\Psi + \mathrm{i}\hbar c\gamma^i\partial_i\Psi = mc^2\Psi - \mathrm{i}mc\omega\gamma^t\gamma^i x^i\Psi.$$

Assume that $\omega > 0$.

(a) Let's begin by diagonalizing H. Letting $\Psi = (\psi_1, \psi_2)$, $a_\pm = (a_x \mp \mathrm{i}a_y)/\sqrt{2}$ where $a_{x,y}$ are the usual annihilation operators for the non-relativistic harmonic oscillator, show that, if $\epsilon \equiv E/mc^2$ is a dimensionless measure of the eigenvalue, and $\xi \equiv \hbar\omega/mc^2$ is a dimensionless measure of the strength of the oscillation energy:

$$(\epsilon - 1)\,\psi_1 = 2\mathrm{i}\sqrt{\xi}a_-^\dagger\psi_2,$$
$$(\epsilon + 1)\psi_2 = -2\mathrm{i}\sqrt{\xi}a_-\psi_1.$$

(b) Conclude that the energy spectrum is

$$\epsilon = \pm\sqrt{1 + 4\xi n}, \quad n = 0, 1, 2, \ldots$$

What are the associated eigenvectors? Explain why there is always one more ψ_2 excitation than ψ_1.

(c) Suppose that we start the system in a state

$$\Psi = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}.$$

Describe the evolution of this system in time. Show that the spin couples to "orbital" (harmonic oscillator) degrees of freedom, and that the system oscillates between $\psi_{1,2}$. Is there physical intuition for why the system can only pass between these two states?[1]

(d) Consider adding a magnetic field B, which comes from the vector potential $\mathbf{A} = (B/2)(y, -x)$. Show that the effective equations of part (a) are *nearly unchanged*, except for a shift in the value of ω to ω_{eff}. Show that the effective value $\omega_{\text{eff}} = 0$ for some critical $B = B_\text{c}$. Describe the eigenvalue spectrum for any value of B; in particular, what happens at $B = B_\text{c}$?

(e) What happens to the chirality of the excitations as B passes through B_c?[2]

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[1] Think about angular momentum.

[2] Remember, the chirality of the excitations is related to the angular momentum. You don't need to show this, but $a_+^\dagger$ and $a_-^\dagger$ create excitations of opposite chirality.