Entanglement Entropy of Free Fermions

Consider a Hamiltonian of free fermions on sites labeled $i = 1, \ldots, N$, with arbitrary hopping and on-site energies:

$$H = -\sum t_{jk} c_j^\dagger c_k.$$

Assume $t_{jk} = \bar{t}_{kj}$. Let $|\text{gs}\rangle$ be the ground state of this system. Denote $M \subset \{1, \ldots, N\}$ as some subset of the sites, and let P_{ij} be a projection matrix onto sites in M. Our goal is to prove that there is a very efficient algorithm for computing the entanglement entropy of region M, $S(M)$, in terms of the matrix

$$A_{IJ} \equiv \langle \text{gs}|c_I^\dagger c_J|\text{gs}\rangle.$$

Here I only runs over indices in set M.

To begin, it will help us to diagonalize H. Let

$$H = \sum_\alpha E_\alpha d_\alpha^\dagger d_\alpha$$

and

$$d_\alpha = U_{\alpha i} c_i$$

(summation convention employed) be the transformation which diagonalizes the Hamiltonian. As is usual for a system with fermions, the ground state of the system is

$$|\text{gs}\rangle = \prod_{E_\alpha < 0} d_\alpha^\dagger |\text{vac}\rangle$$

(We are taking the Fermi energy to be 0.) Let $\Pi_{\alpha\beta}$ be a projection matrix onto states with $E_\alpha > 0$.

(a) Find an expression for the matrix A_{ij}.

(b) Using the properties of the ground state, and Wick's theorem, explain why the reduced density matrix of region M,

$$\rho(M) = \operatorname*{tr}_{N-M} |\text{gs}\rangle\langle \text{gs}|,$$

must be of the form

$$\rho(M) = \exp\left[-B_{IJ} c_I^\dagger c_J\right].$$

(c) Show that the eigenvectors of A and B are the same, and relate their eigenvalues.

(d) Conclude that

$$S(M) = \operatorname{tr}\left[-A\log A - (1-A)\log(1-A)\right].$$

The entanglement entropy can often be used as a very useful order parameter for describing phases of quantum field theories. Even when it cannot be used as a true order parameter, it can often nonetheless be a very useful way of understanding transitions between two different regimes of a field theory.

For example, consider the 1 dimensional Anderson model on a circular lattice of N sites with on-site disorder, which has a Hamiltonian given by

$$H = -\sum_{i=1}^{N} \left(c_i^\dagger c_{i+1} + c_i^\dagger c_{i-1} + \epsilon_i c_i^\dagger c_i \right)$$

where $N+1 \sim 1$, and ϵ_i are iid zero-mean Gaussian random variables with variance σ. It is known that for any $\sigma > 0$, all eigenvectors of H are localized in the large N limit. However, as $\sigma \to 0$, on "small length scales" the model should look free. One can parameterize how free this model looks by a correlation length ξ, which is known to scale as $\xi \sim \sigma^{-2}$. Let the region M consist of lattice sites $1, \ldots, L$.

(e) Heuristically sketch what you think $S(L)$ should look like, as a function of L. Take the limit $N \to \infty$. Whenever appropriate, make sure you have the correct pre-factors in front of leading order terms.

(f) Write some numerics to simulate $S(L)$ on random instances of the Anderson model. Plot $S(L)$, averaged over many realizations of disorder, for various values of σ. Do you recover the prediction of the previous part?

(g) Recover the scaling $\xi \sim \sigma^2$ from numerical simulations of this model, using entanglement entropy.