Gross-Neveu Model

The Gross-Neveu model is to a theory of N massless interacting Dirac fermions in $d = 1 + 1$ spacetime dimensions under the following Lagrangian:

$$\mathcal{L} = -\mathrm{i}\bar{\psi}_i \partial\!\!\!/ \psi_i + \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2.$$

Let's begin with a few basic observations:

(a) Find a symmetry of $\mathcal{L}$ that explains why no mass term will appear in $\mathcal{L}$, to all orders of perturbation theory.

(b) Show that in $d = 2$, the interaction in $\mathcal{L}$ is marginal.

(c) Show that the matrices $\gamma^0 = \sigma^2$ and $\gamma^1 = \mathrm{i}\sigma^1$, where σ^i are Pauli matrices, satisfy the appropriate identities for γ matrices in $d = 2$. These will be useful in calculations!

(d) Directly compute the 1-loop β function for g, and show it is given by

$$\beta(g) = -\frac{Ng^2}{2\pi}.$$

Therefore, this is an asymptotically free theory.[1]

This model also exhibits spontaneous symmetry breaking! While we found that mass terms cannot appear in $\mathcal{L}$ through perturbation theory, another approach will demonstrate that quantum fluctuations provide ψ_i with an effective mass term. Let's see how this works:

(e) Show that

$$\int \exp\left[\mathrm{i}\int\left[-\mathrm{i}\bar{\psi}_i\partial\!\!\!/\psi_i - \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2\right]\mathrm{d}^2x\right]\mathrm{D}\psi_i\mathrm{D}\bar{\psi}_i$$
$$= \int \exp\left[\mathrm{i}\int\left[-\mathrm{i}\bar{\psi}_i\partial\!\!\!/\psi_i - \lambda\bar{\psi}_i\psi_i - \frac{\lambda^2}{2g^2}\right]\mathrm{d}^2x\right]\mathrm{D}\lambda\mathrm{D}\bar{\psi}_i\mathrm{D}\psi_i.$$

Here λ is an auxiliary scalar field.

(f) Integrate out the fermionic fields to find an expression for the effective potential for λ^2, to whatever order is necessary to find a potential with a minimum for any N. Use any renormalization scheme you'd like to deal with divergences.

(g) Verify that the β function found in part (d) is correct.

(h) Show that λ obtains a VEV given by

$$|\langle\lambda\rangle| = \lambda_0 \mathrm{e}^{-\pi/Ng^2},$$

where the particular expression for λ_0 will depend on the renormalization scheme (although the scaling will not).

[1]The Gross in Gross-Neveu model refers to David J. Gross, who won the Nobel prize in Physics in 2004 (along with others) for his discovery of field theories with asymptotic freedom.

Looking at the expression of part (e), we see that the fermions have obtained a mass. This is a sign that some symmetries in quantum field theory can be broken in an entirely non-perturbative way. Unfortunately, in general, non-perturbative QFT is quite challenging.