Airfoils

In this problem, we will explore how an airfoil, such as the wing of an airplane, generates lift by using a simple analytic model. In this problem, you can assume that air is an incompressible and irrotational fluid with density ρ.

In order to model our airplane wing, we will use a conformal mapping to generate a shape which "looks like" an airplane wing. The conformal mapping of choice is called the **Zhukovsky transformation**:

$$f(z) = z + \frac{b^2}{z}.$$

(a) Consider the contour

$$C(\theta) \equiv a\mathrm{e}^{\mathrm{i}\theta} - z_0$$

where $z_0 = \epsilon \mathrm{e}^{-\mathrm{i}\delta}$, with parameters a, ϵ, and δ real, $a > b$, and $\theta \in [0, 2\pi)$. Now, perform the conformal mapping and look at the contour $f(C(\theta))$. Sketch this new contour, and show that it "looks like" an airplane wing.

As usual, we will exploit the conformal mapping to calculate things in the much simpler pre-transform reference frame. The following complex potential represents flow around the circle, with fluid flow of speed v_0 far from the airplane:

$$w = v_0 \left(\mathrm{e}^{-\mathrm{i}\alpha}(z - z_0) + \frac{a^2 \mathrm{e}^{\mathrm{i}\alpha}}{z - z_0} \right) + \frac{\mathrm{i}\Gamma}{2\pi} \log\left((z - z_0)\mathrm{e}^{-\mathrm{i}\alpha} \right).$$

Let us also assume that this airplane wing has a "large" length L.

(b) What is the physical interpretation of Γ? What about α?

(c) Use the requirement that the velocity must be finite at all points along the airfoil to obtain

$$\Gamma = 4\pi v_0 a \sin(\alpha + \beta),$$

where β is the angle made between the real axis and the line connecting z_0 to the left intersection point of $C(\theta)$ with the real axis.

(d) Use Blasius' Theorem in the pre-transform coordinates to conclude that the lift on the airplane wing is given by

$$F_{\text{lift}} = 4\pi \rho a L v_0^2 \cos\alpha \sin(\alpha + \beta).$$