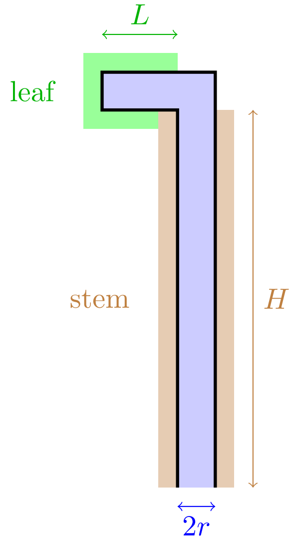
Leaf Sizes

In this problem, we will make theoretical estimates for the range of possible leaf sizes in (angiosperm) trees. To do this, we will suppose that a tree "engineers" leaves to satisfy some basic constraints imposed by the physics of sugar transport in the tree. Let's begin by recalling that sugars are made via photosynthesis in the leaves of a tree, and must then be transported throughout the tree down the stem. We sketch the relevant parts of the tree below: sugars are transported in a "phloem tube" of radius r, which extends down the leaf of length L, and down the stem of height H. In this problem, we will use the circuit analogy for fluid flow. There are two effective resistors, due to flow down the stem, and flow across a porous membrane.



(a) It is reasonable to approximate that the resistance in the leaf (which has a fairly small length L, compared to H), is dominated by the transport of sugary fluid across the membrane. Explain why it is reasonable that this resistance scales as $R^{-1} = A\mathcal{L}$, where $\mathcal{L}$ is a coefficient of proportionality, and A is the area of the membrane.

(b) Determine the fluid resistance of the stem. If the sugary water's viscosity is η, show that the fluid flow through the phloem tube is given by

$$u = \frac{2r^2\mathcal{L}L}{r^3 + 16\eta\mathcal{L}LH}\Delta P,$$

where ΔP is the pressure gradient.

(c) Assume H and L are fixed, and r is variable. Show that the maximum value of u occurs when $r \sim (LH)^{1/3}$. What is the coefficient of proportionality? This is, interestingly, quite well-documented in plants. You may assume that r takes this value for the remainder of the problem.

(d) Now, argue that there is some scale of L, $L_{\max}$, for which the speed does not get any larger. What is this scale?

(e) One might think that the plant might as well make leaf sizes very small. However, this will make the fluid flow very slow, and one might worry that diffusion will dominate on these small scales. If the diffusion constant for the sugars is D, and the intracellular distance is d, we would like for convective sugar transport to be faster than the intracellular diffusion speed. Show that requiring that the convective speed be larger than this diffusive speed gives a lower bound to L. You may assume that the diffusive speed is very small, when finding your answer.

(f) Using the numerical values of $\Delta P = 10^6$ Pa, $\eta \approx 5 \times 10^{-3}$ Pa $\cdot$ s, $\mathcal{L} \approx 5 \times 10^{-14}$ m/s $\cdot$ Pa, $d \approx 4 \times 10^{-5}$ m, and $D \approx 5 \times 10^{-10}$ m^2/s, calculate $L_{\text{max,min}}$ for a tree which is 5 m tall. Leaves are known to span sizes anywhere from about 1 mm to 1 m. Does your calculation mesh with this result? Speculate briefly on a discrepancy, if it arises.