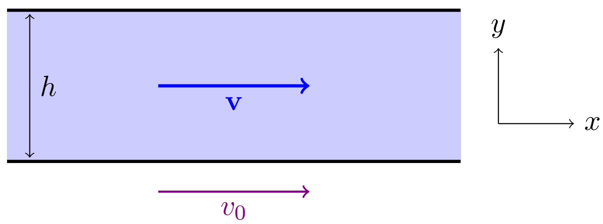
Lubrication

In this problem, we will explore how viscous fluids flowing through very small gaps between two surfaces, which are moving relative to one another, can greatly reduce the resistive force of such motion. This is called **lubrication**, and was mathematically explored first by Reynolds in 1886. Lubricants are now used frequently, everywhere from oil lubricants inside of car engines to air, acting as a lubricant to allow a computer's disk drive to spin rapidly without damage.

To begin with, consider the following simple fluid flow: an incompressible fluid of density ρ and viscosity η flows between two infinite parallel plates, separated by a distance h. The plate at $y = 0$ moves with a velocity v_0, and the plate at $y = h$ is stationary. Assume that the only non-zero component of velocity is of the form $v_x(y)$.



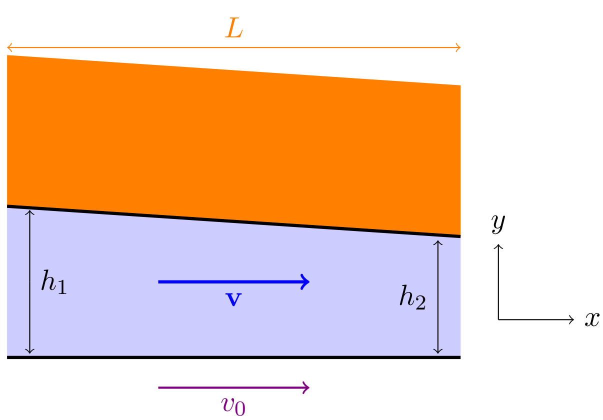
(a) Show that the pressure is of the form $P = P(x)$.

(b) Solve the Navier-Stokes equation exactly to find $v_x(y)$ for the appropriate boundary conditions. Leave the answer in terms of an unknown pressure gradient, $\mathrm{d}P/\mathrm{d}x$.

Now, assume that we have the following set-up. A fixed block has a length L and width w, and is tilted with respect to a moving infinite plate below it, in such a way that the distance between the two plates is given by

$$h(x) = h_1 + (h_2 - h_1)\frac{x}{L}.$$

The pressure to the left and right of the block is given by $P = P_0$. The plate below has velocity v_0. Assume that $h \ll w, L$, so that approximately the fluid flow can approximately be treated as flow between two "parallel" infinite plates, for any given value of x.



(c) Use your solution from the previous part, as well as the stated approximations and all appropriate physical conservation laws, to show that

$$6v_0\eta\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(h^3\frac{\mathrm{d}P}{\mathrm{d}x}\right).$$

This is often called Reynolds' lubrication equation.

(d) In particular, given the form of $h(x)$ above, and the boundary conditions on the pressure $P(x)$, show that

$$P - P_0 = \frac{6\eta v_0 L}{h_1 - h_2}\left(\frac{1}{h(x)} - \frac{1}{h_1}\right)\left(1 - \frac{h_1 h_2}{h_1 + h_2}\left(\frac{1}{h(x)} + \frac{1}{h_1}\right)\right).$$

(e) Suppose that the stationary block is a cube of side length $L = w = 0.1$ m, and weighs 10 kg (with the mass uniformly distributed), and thus imparts a force of about 100 N onto the fluid in the gap between the block and plate. Suppose that $h_2 = 0.9h_1$, and $h_1 \approx 10^{-4}$ m, and the fluid in between the objects is a lubricating oil with $\eta \sim 10^{-1}$ Pa/s. If the weight of the block is balanced by the pressure from the viscous fluid, determine the order of magnitude of v_0.

(f) Normal friction forces are of the same order of magnitude as weight. Compare this to the rough size of the resistive force due to the viscous fluid, and thus justify the use of lubricants as a way to make dissipative forces significantly smaller.