Orr-Sommerfeld Equation

The Orr-Sommerfeld equation describes the perturbations around the pressure-driven (Poiseuille) viscous flow of fluid in a channel in two dimensions. Consider a fluid with viscosity η, flowing with velocity $u_x(y)$ down a channel which extends infinitely in the x-direction, and between $|y| \le h/2$. This flow is driven by a pressure gradient, which we take to be a constant: $\mathrm{d}P/\mathrm{d}x \equiv -\alpha$ (take $\alpha > 0$). The flow is independent of x and t.

(a) What is the solution to the background problem? You probably know this already, and can just write it down.

(b) Now, suppose that we perturb this Couette flow with a perturbation of the form

$$\delta u_x = \delta u_x(y)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t},$$
$$\delta u_y = \delta u_y(y)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t},$$
$$\delta P = \delta P(y)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}.$$

Show that, after appropriate non-dimensionalizations, the incompressible Navier-Stokes equations reduce to the following single differential equation, where $\mathcal{R}$ is the Reynolds number of the flow (written in terms of the usual stream function ψ, whose Laplacian is the vorticity):

$$\mathrm{i}(k(1-y^2) - \omega)(\partial_y^2\psi - \psi) = \mathcal{R}^{-1}(\partial_y^2 - 1)^2\psi - 2\mathrm{i}k\psi$$

Show that the boundary conditions on the stream function ψ are that ψ and $\partial_y\psi$ vanish at the boundaries. This equation is called the **Orr-Sommerfeld equation**.

(c) In 1972, it was shown that spectral methods are a very accurate and easy way of computing the eigenfrequencies ω. We're looking for the onset of an instability (to turbulence). Use the fourth-order spectral method based on Chebyshev polynomials in the interval $[-1, 1]$ with the clamped boundary conditions relevant for this problem. Write some code to calculate the values of ω. Then, explore the eigenfrequencies for various discretizations (with N points). You should find that the critical value of $\mathcal{R} \approx 5772$, and the critical value of $k \approx 1.02$, at which an instability occurs.