Poiseuille Flow in Complicated Pipes

Suppose that we have some pipe which is hollow, and is described by the region $(x, y) \in \Sigma$, $0 \le z \le L$, where Σ denotes some arbitrary (connected) region of the two-dimensional plane. Suppose that the area of Σ is A, and its perimeter is S. If we fill this pipe with a fluid of viscosity η, and apply a pressure gradient ΔP across the ends of the pipe, if L is large compared to S and $\sqrt{A}$, we expect for some general Poiseuille flow to be set up, where there is a constant pressure gradient, and velocity flow only in the z direction. Analogously to the simple exactly solvable cases, we expect

$$\nabla^2 v_z = -\frac{\Delta P}{\eta L}.$$

As usual, we'll impose the no-slip boundary conditions on v_z at the boundary of Σ.

For some complicated shape, we cannot solve this equation analytically. However, often times we are not interested in the complicated details of the flow, but only in the *mass flow rate* Q through the pipe:

$$Q = \int_\Sigma \mathrm{d}x\mathrm{d}y\ v_z.$$

One often finds that

$$\Delta P = QR,$$

where R is a constant called the hydraulic resistance. It turns out that through a simple argument that we employ in this problem, the scaling of R can be understood without a detailed understanding of the geometry.

(a) Let f_n denote the eigenfunctions of the Laplacian in the domain Σ with Dirichlet boundary conditions: i.e. $\nabla^2 f_n = -\lambda_n f_n$ for some λ_n. Denote the functions:

$$v_z = \sum_n a_n f_n,$$
$$1 = \sum_n b_n f_n$$

where v_z is the solution to the Poiseuille flow equation. Relate a_n to b_n.

(b) By whatever argument you wish, determine the scaling of the λ_n in terms of A and S. Conclude that for a generic shape, we should find that the hydraulic resistance is

$$R \sim \frac{\eta L S^2}{A^3}$$

where the only thing we are missing in the above formula is some O(1) proportionality constant.