Vortex Filaments in Superfluids

A superfluid is a *quantum fluid* with no viscosity and no vorticity (it is a potential flow). These fascinating fluids are the subject of much interest in condensed matter physics as macroscopic quantum systems. Typical superfluids are various types of helium at very low temperatures $T \sim 1$ K.

However, superfluids have an instability when they are subjected to a global rotation at angular velocity Ω. This problem will model this instability. In a non-rotating reference frame, the energy of the superfluid is given by

$$E = E' - L\Omega$$

where L is the angular momentum of the fluid and E' is the energy of the fluid, as seen in the rotating frame. Now, if Ω is large, is there some potential flow configuration with a small E' but large L? Although the creation of any vorticity – required for the fluid to have a net angular momentum, would violate the potential flow assumption, the superfluid gets around this problem by the creation of a **vortex filament** – a very small region where the superfluid approximation breaks down. Within the filament, there is no longer superfluidity, and thus vorticity can be present.

To model such a vortex filament, we will assume that we have a superfluid in a cylindrical container of height H and radius R. Assume that the filament runs along the central axis of the cylinder, and that the superfluid has density ρ and that there are no external pressures.

(a) Explain why we expect the superfluid velocity to be of the form

$$\mathbf{v} = \frac{\gamma}{r}\hat{\boldsymbol{\phi}},$$

where r is the distance from the center of the cylinder. Note that this formula breaks down for $r \sim a$, where a is a microscopic scale corresponding to the interatomic spacing in the superfluid.

(b) Quantum mechanically, there is a constraint on the allowed values of γ. Essentially the phase of the quantum wave function of the superfluid must not change as it winds around the vortex, and this gives the constraint

$$m \oint \mathrm{d}\mathbf{s} \cdot \mathbf{v} = 2\pi\hbar n$$

for integer n. Here $\hbar$ is the reduced Planck's constant and m refers to the mass of the constituent particles. Argue that so long as the path of integration winds once around the filament, the choice of path does not matter, and then constrain the values of γ.

(c) For each given $n \neq 0$, compute the energy E_n in the rotating frame of the vortex filament. Which filament is most likely to form? Show that the filament will form when

$$\Omega > \frac{\hbar}{mR^2} \log \frac{R}{a}.$$

(d) Reasonable scales for this problem are $a \sim 10^{-9}$ m, $m \sim 10^{-26}$ kg, and $\hbar \approx 10^{-34}$ J $\cdot$ s. Do you think vortex filaments can be observed "easily" in a lab? Would it be reasonable to prepare a superfluid with no vortex filaments?

(e) What do you think happens as Ω is increased?