Vortex Knots

In this problem, we will predict a fascinating phenomenon called a **vortex knot** – a closed loop of vorticity in space. Such vortex knots may be relevant in turbulent fluid dynamics.

To begin, let us recall that the velocity of a vortex filament is given by

$$\partial_t \mathbf{r}(s_0) = \frac{\Gamma}{4\pi}\int \mathrm{d}s \frac{\partial_s \mathbf{r} \times (\mathbf{r}(s_0) - \mathbf{r}(s))}{|\mathbf{r}(s_0) - \mathbf{r}(s)|^3}$$

where Γ is the circulation of the vortex, and $\mathbf{r}(s)$ describes the position of the filament at "position" s. Feel free to normalize s so that $|\partial_s \mathbf{r}| = 1$.

It is "impossible" to find nontrivial solutions to the Biot-Savart equation above. However, there are approximations for which interesting solutions do exist. We will consider one called LIA (localized induction approximation), in which we approximate that

$$\mathbf{r}(s) \approx \mathbf{r}(s_0) + \partial_s \mathbf{r}(s_0)(s - s_0) + \frac{1}{2}\partial_s^2 \mathbf{r}(s_0)(s - s_0)^2,$$

and neglect all higher order terms.

(a) Show that under LIA, up to a log-divergence term which can be absorbed into an overall constant C, the Biot-Savart law reduces to

$$\partial_t \mathbf{r} = C\partial_s \mathbf{r} \times \partial_s^2 \mathbf{r}.$$

(b) As we'll be looking for knot solutions, it will be convenient to switch to cylindrical coordinates (r, ϕ, z). Show that in these coordinates:

$$\begin{aligned}
\frac{1}{C}\partial_t r &= r\partial_s\phi\partial_s^2 z - 2\partial_s r\partial_s\phi\partial_s z - r\partial_s^2\phi\partial_s z, \\
\frac{1}{C}\partial_t \phi &= \frac{\partial_s^2 r\partial_s z - \partial_s r\partial_s^2 z}{r} - (\partial_s\phi)^2\partial_s z, \\
\frac{1}{C}\partial_t z &= 2(\partial_s r)^2\partial_s\phi + r\partial_s r\partial_s^2\phi + r^2(\partial_s\phi)^3 - r\partial_s^2 r\partial_s\phi.
\end{aligned}$$

(c) Verify that

$$\begin{aligned}
r &= R, \\
\phi &= \frac{s}{R}, \\
z &= \frac{Ct}{R}
\end{aligned}$$

satisfy the equations of part (b), if R is a constant. What does this solution physically correspond to, and what does R mean?

(d) Now, linearize the full equations of motion about this solution. Show that you can obtain

$$\begin{aligned}
\frac{1}{C}\partial_t r_1 &= \partial_s^2 z_1, \\
\frac{1}{C}\partial_t \phi_1 &= -\frac{\partial_s z_1}{R^2}, \\
\frac{1}{C}\partial_t z_1 &= -\partial_s^2 r_1 - \frac{r_1}{R^2}.
\end{aligned}$$

(e) Let us look for soliton (traveling wave) solutions of the form $\mathbf{r}(s,t) = \mathbf{r}(s-ut)$ where u corresponds to the wave velocity. Imposing the conditions that r_1 is a periodic function in s with some period L, show that you find

$$r_1 \sim \cos\left(\beta + \frac{2q\pi}{L}(s-ut)\right)$$

(since these are linear equations, the overall constant is undetermined) for an arbitrary constant β, along with the wave speed

$$u = C\sqrt{\left(\frac{2q\pi}{L}\right)^2 - \frac{1}{R^2}}.$$

Also, find ϕ_1 and z_1 as functions of $s-ut$. These are the promised vortex knots!

(f) Explain why the fact that ϕ is an angular coordinate restricts us to take $L = 2\pi Rp$, for $p = 1, 2, \ldots$.

(g) Explain why only solutions with $q > p$ are stable.

(h) Show that the vortex knot solutions you have found are *toroidal knots* in that the perturbed filament lives on an elliptical torus. The numbers q and p correspond to windings of the knot around the cycles (the "inner/outer circles") of the torus.

(i) Topologically, the knot corresponding to q and p is the same as the knot with those integers reversed. However, we have just found one of these knots is stable, and one is unstable. Give a simple explanation for why this fact should not be surprising.