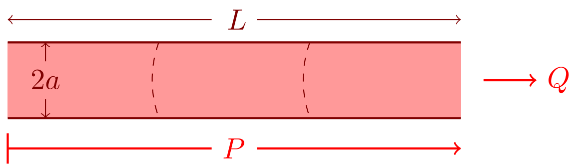
Circulatory System as a Circuit

The flow of blood inside of an animal comprises its circulatory system. The hydrodynamics of blood is a very tricky subject to study, both because of the difficulty of doing experiments and the highly nonlinear and non-Newtonian nature of blood. Nonetheless, some basic results of interest can be obtained by the construction of simple theoretical models. For this problem, assume blood is an incompressible fluid with density ρ. Our first goal is to argue that some aspects of the circulatory system can be modeled by electrical circuits. After that, we will use some very simple but beautiful arguments to derive a subset of an incredibly beautiful set of generic scaling laws governing biological organisms, based entirely on simple arguments related to the flow of blood in the circulatory system.

The simplest model of a blood vessel consists of a rigid tube of radius a and length L filled with flowing blood with Newtonian viscosity coefficient η. Suppose that the pressure difference across the vessel is P. Assume all fluid flow is oriented along the blood vessel (no radial or angular components), and assume the flow is static.



(a) Find an expression for the velocity of the blood inside of the vessel.

(b) Show that the volume flow rate Q is related to the pressure drop P by a linear relation, $P = RQ$, where the coefficient R is given by

$$R = \frac{8\eta L}{\pi a^4}.$$

We will call R the "resistance" of the blood vessel, later in the problem.

Now, suppose the front end of the blood vessel connects to the heart. A simple model of the heart consists of a chamber with volume V, and compressibility coefficient $\kappa > 0$, such that if its equilibrium volume is V_0:

$$V - V_0 = -\kappa P.$$

Muscles help cause the heart to contract, forcing the flow of blood, as depicted in the diagram below. In this model, the heart pumps blood into a simple blood vessel of the type studied just above.

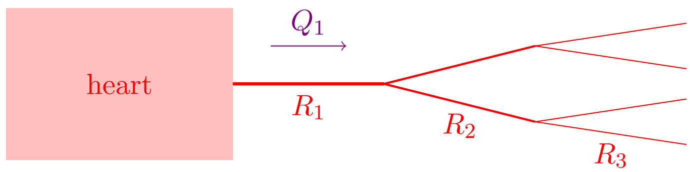
(c) Suppose that the heart is "driven" by a pressure $\cdots + G\delta(t) + G\delta(t-\tau) + \cdots$. This corresponds to the muscle which causes heartbeats. What is $\langle Q \rangle$, the average flow rate pumped by the heart? Ignore the mechanisms by which blood returns to the heart, and leaves the blood vessel.[1]

(d) Discuss how the circulatory system of an animal can be related to an electrical circuit: i.e., what represents current, voltage etc.? Discuss what circuit elements represent a blood vessel and the heart.

Now, let's turn to biology. Again, for simplicity only considering the part of the circulatory system which sends blood away from the heart, and not back to the heart, the circulatory system of a typical

[1] Think about how $V - V_0$ relates to Q.

animal could be modeled as follows: there is a heart, which pumps according to the simple pulses given in part (c), and which sends blood flowing down a vessel with resistance R_1, at an average flow rate Q_1. Then this vessel splits into n vessels with resistance R_2; subsequently, those vessels split into n vessels with resistance R_3, etc..., so in general the resistance of each vessel in the k^{th} branch is R_k.



(e) By an analogy to circuits, show that the effective resistance seen by the heart is[2]

$$R_{\text{eff}} = \sum_k \frac{R_k}{n^k}.$$

Of course, to get any farther, we need to determine what R_k is, and we will do this by arguing what L_k and a_k are: i.e., the length and radius of the vessels, respectively.

(f) Assume that the blood in one vessel is approximately destined for all locations in the body which are within a distance of L from the center of the blood vessel. If, as each blood vessel splits into n smaller vessels, these smaller vessels are also destined for the same volume, determine the ratio L_{k+1}/L_k.

(g) Assuming that the average velocity of the blood is constant in all blood vessels, determine the ratio r_{k+1}/r_k.

Our next assumption will be that this branching process stops at a series of very tiny tubes, which we call the **capillaries**. Each capillary has a resistance of R_c, which we'll approximate is universal for each biological organism (with a circulatory system). Let N_c be the number of capillaries in the organism (this will certainly be an organism dependent quantity!).

(h) Combining the results of the previous 3 parts, the assumptions above, and making all appropriate approximations, show that

$$R_{\text{eff}} \approx \frac{R_\text{c}}{N_\text{c}}.$$

Now that we've developed the above results, we can finally start to answer some interesting questions. For starters, we've assumed that the blood's velocity does not change as it flows through the circulatory system. Let's think about the consequences of this. Since the blood is delivering oxygen to the cells, we should roughly expect that each cell requires some fixed amount of blood flow per unit time to sustain its oxygen needs. Also, recall that a cell is made up of mostly water.

(i) Using the above argument to get you started, conclude that the mass M of our organism should be proportional to the total volume V_{circ} of all of the blood vessels in its circulatory system.

(j) Since each capillary feeds the cells with some fixed amount of oxygen per unit time, it is natural to assume that the metabolic rate (the rate at which energy can be spent by the cells) of the total organism, B, is proportional to N_c. Show that

$$B \sim M^{3/4}.$$

This result is *quantitatively* verified by numerous experimental results in biology.

[2] It is probably easiest to do this by using the fact that the power dissipated in a circuit, if the current flowing down resistor R_1 is I_1, is given by $P = I_1^2 R_{\text{eff}}$.

(k) Suppose that each cell will expend an energy E before it must die. Argue that the time scales of an organism, such as its lifetime, scale as $\tau \sim M^{1/4}$.