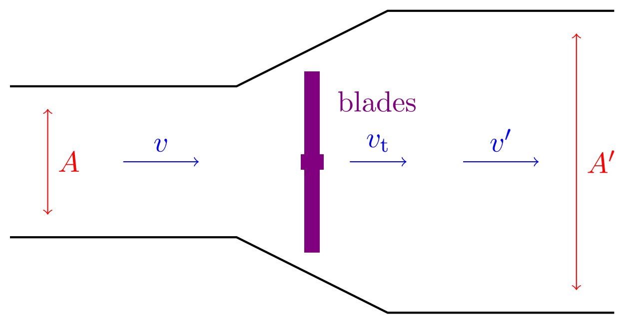
Efficiency of a Wind Turbine

In this problem, we will determine the maximal possible power that can be extracted from a wind turbine. We imagine the following setup. Consider a tube of cross-sectional area A, with incident gas velocity v. We place some turbine blades in this tube, and then open up the tube to some area A', as shown below:



For simplicity, assume that the air is described by an incompressible fluid of density $\rho \approx 1.25$ kg/m^3.[1] Let A_t denote the "effective area" of the turbine, and v_t denote the effective velocity of the fluid in the turbine.

(a) Use mass conservation to fix v_t and the exit velocity, v'.

(b) Now, compute the power of the wind turbine by computing the rate at which kinetic energy is lost to the turbine blades – i.e., compute the differences in the energy fluxes into and out of the turbine entrance and exit.

(c) Compute the power of the turbine in a different way – using momentum conservation. Be careful – the force we are extracted is applied at the turbine blades! Show that comparing to the previous part constrains the values of A_t and v_t.

(d) Optimize the geometry of the turbine, and conclude that the maximal possible power we can extract from the air is

$$P = \frac{4}{9}\rho A v^3.$$

(e) Argue that the "ideal" amount of power we can extract is given by

$$P_\mathrm{ideal} = \frac{1}{2}A_\mathrm{t}\rho v^3.$$

The efficiency e of the turbine can then be defined as P/P_ideal. What is the maximal value of e?

(f) Typical wind speeds in a "favorable" weather area for wind power are around 10 m/s. A typical family in the USA uses about 1 kW of power. Estimate the size of a wind turbine required to provide power for this family. The largest turbines in the world have blades of radius ~ 60 m. How many families could such a wind turbine support?

[1] Thus, our results are also perfectly good at describing "ocean turbines" which use water instead of air. Of course in this case, $\rho \approx 1000$ kg/m^3.