continuum mechanics → viscous fluids ★★★★

# Hele-Shaw Viscous Fingering

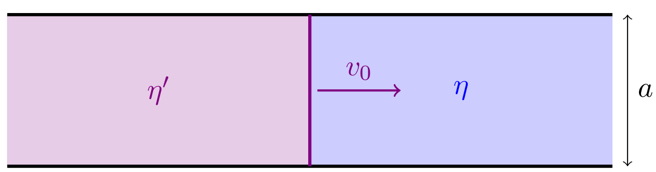
Consider a fluid such as water, with viscosity $\eta$ and mass density $\rho$, contained in between two plates separated by a distance $a$ which is small enough that the Reynolds number is very low.

(a) Using the values $\rho \approx 10^3$ kg/m$^3$ and $\eta \approx 10^{-3}$ for water, how small does $a$ need to be, assuming that the velocities of interest are no larger than 0.1 m/s?

(b) Now, suppose that the Reynolds number is small so that the fluid is in the limit of creeping flow. If $\langle v_i \rangle$ is the velocity of the fluid, averaged over the thickness $a$ of the dish, show that $\langle v_i \rangle \sim \partial_i P$, and find the coefficient of proportionality.

(c) Show that incompressibility of the fluid implies that the pressure obeys Laplace's equation in 2 dimensions.

Now, suppose that we are pushing a new fluid of viscosity $\eta'$ into the dish, such that the velocity of the interface between the two fluids is $v_0$:




Denote the $x$ direction to be the direction of motion of the interface: i.e., the interface's position is given by $x(t) = v_0 t$.

(d) Find the pressure $P$ (up to an overall constant) and show it is a function only of $x - v_0 t$.

Now, suppose that there is a small perturbation in the perpendicular $y$ direction, such that the interface is now described by the surface

$$x(y,t) = v_0 t + \delta(t)\mathrm{e}^{\mathrm{i}ky}$$

with $\delta(t)$ a small parameter. Due to the interface, we will now expect see perturbations to $P$ on either side of the interface at order $\delta$:

$$P - P_{\mathrm{orig}} = \delta(t)c(x,t)\mathrm{e}^{\mathrm{i}ky}$$

with $P_{\mathrm{orig}}$ the pressure found in part (d), but now measured as a function of the $x$-displacement from the interface. Note we will allow $c$ to be complex to take into account a phase difference with the interface perturbation.

(e) Find an expression for $c(x,t)$ up to an as-of-yet undetermined function of $t$.

(f) Use the boundary condition on fluid velocity normal to the interface to find an equation relating the value of $c$ just to the left of the interface and just to the right.

(g) Suppose that the surface tension of the interface is $\sigma$. Assuming small curvature of the surface, completely determine the function $c(x,t)$.

(h) Show that

$$\dot{\delta} = \frac{ka^2}{12(\eta + \eta')} \left[ \frac{12 v_0}{a^2} \left( \eta - \eta' \right) - \sigma k^2 \right] \delta.$$

(i) Under what circumstances will the interface be unstable? If so, find the critical wavelength $\lambda_c$ at which the interface becomes unstable.

(j) Explain in words why the fingering process occurs.