1D Forest Fire Model

The forest fire model is one of the earliest examples of a model with self-organized criticality – a model with robust scale invariance that does not require the tuning of a critical parameter. In 1D, the forest fire model has an exact solution in the critical regime. The goal of this problem will be to find this exact solution and determine the critical exponents exactly.

The forest fire model goes as follows. Let us consider a 1D lattice with variables $n_i \in \mathbb{Z}_2$, for $i \in \mathbb{Z}$, with $n_i = 0$ denoting a site where there is no tree, and $n_i = 1$ denoting a site where there is a tree. Now, at each time step, with some probably $f \ll 1$, a lightning strike hits site i (for each i). If there is a tree at site i ($n_i = 1$), then the lightning strike starts a forest fire, and the "forest" around site i will burn down: i.e., the string of 1s including i will all become 0. If there is not a tree at site i, nothing happens. In addition, we then allow for regrowth. With probability p, each site has the chance for regrowth: if $n_i = 0$, then $n_i = 1$ after regrowth; if $n_i = 1$, nothing happens. We will always take $p \gg f$: thus, the forest has a chance to regrow large clusters of trees before they burn again.

Given some initial conditions, after some transient time (not of interest to us), the system should reach some sort of equilibrium. Our goal is to understand what this equilibrium corresponds to. In the equilibrium state, let ρ correspond to the fraction of sites which have trees.

(a) Let $\langle s\rangle$ correspond to the average number of trees destroyed by a given lightning strike. By comparing the number of destroyed trees to the number of grown trees at a given time step, show that

$$\langle s\rangle = \frac{p}{f}\frac{1-\rho}{\rho}.$$

The main question of interest to us is to understand the distribution of forest cluster sizes $\mathrm{P}_{\mathrm{eq}}(s)$ in equilibrium: i.e., what is the probability in equilibrium that a forest cluster (which can be destroyed by lightning) has length s? Based on scaling arguments, *if* this model is critical, we expect that

$$\mathrm{P}_{\mathrm{eq}}(s) \sim s^{-\tau} F\left(\frac{s}{s_{\mathrm{c}}}\right).$$

The cutoff function F is due to the fact that this model is not "truly" critical, but only critical over a very large range of scales, up to the scale s_{c}. You will see why this must be so as we go through the arguments to derive τ.

Our strategy will be as follows. Let us look in the limit as $f \to 0$. In this limit, if we look at a single site, the dynamics should clearly be dominated by tree growth, and the same should be true so long as we look at very small clusters of trees, of length k. Let us denote $\mathrm{P}_k(n_1, \cdots, n_k)$ with the probability that a string of k neighbors has configuration n_1 to n_k in the equilibrium distribution. We'll have to take k "small" (we'll come back to what this means later).

(b) Argue that if the dynamics of small clusters are dominated by forest growth, then with very high probability, a small cluster should only burn when it is part of a very large cluster. Let r_n be the rate at which a cluster of trees of length n burns. Explain why for small n we can approximate:

$$r_n \mathrm{P}_n(1, \cdots, 1) = r_{n-1}\mathrm{P}_{n-1}(1, \cdots, 1) = p(1-\rho).$$

(c) Given our assumptions above, explain why it is reasonable to assume that $\mathrm{P}_k(n_1, \ldots, n_k)$ only depends on $n_1 + \cdots + n_k$ (the number of trees in the cluster).

(d) Find a formula for $\mathrm{P}_k(0)$, assuming we are in equilibrium. Then, use a recursive relationship between $\mathrm{P}_k(m)$ and $\mathrm{P}_k(m-1)$ to derive the following formula:

$$\mathrm{P}_k(m) = \frac{1-\rho}{k-m}.$$

(e) What is the exponent τ in 1D?

(f) Now, looking back at the above argument, give a simple physical/mathematical argument why the 1D forest fire model cannot be "truly" critical (i.e., there must be some cutoff scale s_c at which scale invariance breaks down).

(g) Combine the arguments from all necessary previous parts to find s_c and ρ, to leading order in the large parameter p/f.