Breaking Gauge Symmetry with Fermions

In this problem we will discover and prove the extent to which non-chiral global symmetries can be spontaneously broken in a gauge theory with gauge bosons and n flavors of massive fermions. Suppose that we have the following real time action:[1]

$$S[\overline{\psi}, \psi, A] = \int \mathrm{d}^d x \left[\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mathrm{i}\overline{\psi}^i_\alpha \gamma^\mu \mathrm{D}_\mu \psi^i_\alpha + \mathrm{i}\overline{\psi}^i_\alpha M_{\alpha\beta} \psi^i_\beta \right]$$

The a indices represent the gauge group, the i indices represent the fermion's representation, and the α indices represent fermion flavors. Let us assume that we have diagonalized the mass matrix M and that $M_{\alpha\alpha} > 0$ for each flavor.

(a) Consider an arbitrary correlator $\mathrm{T}\langle \psi^{i_1}_{\alpha_1}(x_1) \cdots \psi^{i_m}_{\alpha_m}(x_m) \overline{\psi}^{j_1}_{\beta_1}(y_1) \cdots \overline{\psi}^{j_m}_{\beta_m}(y_m)\rangle$. By writing this correlator in terms of a path integral over the gauge fields (and ghosts) only, show that it is explicitly invariant under any $U \in \mathrm{U}(n)$ with $[M, U] = 0$.

(b) Describe the subgroup of $\mathrm{U}(n)$ such that $[M, U] = 0$, for an arbitrary mass matrix.

The subgroup of $\mathrm{U}(n)$ described above is what we wish to show cannot be spontaneously broken by quantum effects. It is not enough, however, to simply show invariance of the correlator under a symmetry: we need to show that these correlators are continuous under small perturbations of the mass matrix M.

The proof we will begin below will also be a bit more intuitive if we replace $\psi^i_\alpha(x)$ with a smeared function

$$\psi^{i_k}_{\alpha_k}(x) \to \int \mathrm{d}^d y F_k(y; x) \psi^{i_k}_{\alpha_k}(y)$$
$$\overline{\psi}^{j_k}_{\beta_k}(x) \to \int \mathrm{d}^d y G_k(y; x) \overline{\psi}^{j_k}_{\beta_k}(y)$$

with the functions F_k and G_k normalized:

$$\int \mathrm{d}^d y |F_k(y; x)|^2 = \int \mathrm{d}^d y |G_k(y; x)|^2 = 1.$$

(c) Assuming that we are still working in the basis where M is diagonal, show that $\forall\ \alpha, k$:[2]

$$\left| \int \mathrm{d}^d x \mathrm{d}^d y \left(\gamma^\mu \mathrm{D}_\mu + M_{\alpha\alpha}\right)^{-1}_{xy} F_k(x) G_k(y) \right| \le \frac{1}{M_{\alpha\alpha}}.$$

(d) The upper bound of the previous part now allows us to bound the path integral over the gauge fields from the first part. Show that

$$\left| \langle \mathrm{T}\psi^{i_1}_{\alpha_1}(x_1) \cdots \psi^{i_m}_{\alpha_m}(x_m) \overline{\psi}^{j_1}_{\beta_1}(y_1) \cdots \overline{\psi}^{j_m}_{\beta_m}(y_m)\rangle \right| \le m! \left[\sum_\alpha \frac{1}{M_{\alpha\alpha}} \right]^m .$$

[1] We will neglect the ghost terms in the action as they will not be relevant for this problem.

[2] Begin with the identity $A^{-1} = \int_0^\infty \mathrm{d}z \mathrm{e}^{-zA}$. Then use that $\gamma^\mu \mathrm{D}_\mu$ is anti-Hermitian.

(e) Finally, suppose that we perturb the mass matrix M to $M + \delta$, with δ infinitesimally small. Show that

$$\lim_{\delta_{\alpha\beta}\to 0} \Big| \left\langle \mathrm{T}\psi^{i_1}_{\alpha_1}(x_1)\cdots\psi^{i_m}_{\alpha_m}(x_m)\overline{\psi}^{j_1}_{\beta_1}(y_1)\cdots\overline{\psi}^{j_m}_{\beta_m}(y_m)\right\rangle_{M+\delta} - \left\langle \mathrm{T}\psi^{i_1}_{\alpha_1}(x_1)\cdots\psi^{i_m}_{\alpha_m}(x_m)\overline{\psi}^{j_1}_{\beta_1}(y_1)\cdots\overline{\psi}^{j_m}_{\beta_m}(y_m)\right\rangle_{M} \Big| = 0.$$

thus completing the proof.

More complicated techniques can be used to show, e.g., that parity will not be spontaneously broken. However, since parity also acts on the gauge fields, this proof is a bit more involved.