Electron in Electromagnetic Wave

The electron in 4D is a Dirac fermion of mass m and charge $-e$ under the U(1) electromagnetic gauge symmetry. Consider such an electron under a background classical gauge field of the form

$$A^\mu(x) = a^\mu \mathrm{e}^{\mathrm{i}k^\nu x_\nu}.$$

Assume the gauge $k^\mu a_\mu = 0$.

(a) Show that appropriate "squaring" of the Dirac equation in this classical electromagnetic background field gives

$$\left(\partial^\mu\partial_\mu - m^2 + e^2 A_\mu A^\mu - 2\mathrm{i}eA_\mu\partial^\mu - \frac{\mathrm{i}e}{2}\Sigma^{\mu\nu}F_{\mu\nu}\right)\psi = 0.$$

Note that it is different than if we started with the Klein-Gordon equation, due to the $\Sigma^{\mu\nu}F_{\mu\nu}$ term. This will turn out to have interesting consequences, as we'll see below!

We look for a solution to the above equation of the form

$$\psi_p = \mathrm{e}^{\mathrm{i}p_\mu x^\mu} f(k_\mu x^\mu),$$

with f some arbitrary spinor function.

(b) Use the gauge condition on A_μ and γ matrix identities to show that

$$(\gamma^\mu\gamma^\nu a_\mu k_\nu)^2 = 0.$$

(c) Show that f satisfies a first-order differential equation, and integrate it. By using the relation found in part (b), conclude that

$$\psi_p = \mathrm{e}^{\mathrm{i}(p_\nu x^\nu + \phi)}\left(1 + \frac{e\gamma^\mu\gamma^\nu k_\mu A_\nu}{2k_\alpha p^\alpha}\right)u$$

with u a Dirac spinor. ϕ is some real valued phase function that should show up in your calculation, and it is not important.

It turns out that we should choose u to be the Dirac spinor corresponding to a free electron of momentum p, normalized so that $\overline{u}u = m/p_0$. This can be justified by considering the limit where we are "very far away from the electromagnetic field".

(d) Show that if I is the space/time-averaged intensity of the electromagnetic wave and ω is its frequency, the electron can be thought of as a traveling plane wave with a *reduced mass* given by

$$m_{\mathrm{eff}} = \sqrt{m^2 - \frac{e^4 I}{2\omega^2}}.$$

To do this, consider the average of the operator $\mathrm{i}\partial^\mu - eA^\mu$ on the state ψ_p. This corresponds to the effective momentum of the electron (justify this statement!).