Fermion Number and QED Anomalies

In this problem, we will perform an explicit computation of the QED anomaly in a quantum mechanical example, and show that we exactly recover the result we showed in class. This exercise should give you some physical intuition for what causes the anomaly in a quantum mechanical system.

(a) To begin, show that the Hamiltonian of a quantum mechanical system of massless Dirac fermions in $3+1\mathrm{D}$ is given by

$$H = -\mathrm{i}\int \mathrm{d}^3x \left[\psi_{\mathrm{R}}^\dagger \sigma_i \mathrm{D}_i \psi_{\mathrm{R}} - \psi_{\mathrm{L}}^\dagger \sigma_i \mathrm{D}_i \psi_{\mathrm{L}}\right].$$

(b) Suppose that these fermions are placed in an external field given by $A^0 = A^1 = 0$, $A^2 = Bx^1$, and A^3 a nonzero constant. Show that the Hamiltonian can be exactly solved, and find the energies of the oscillator.[1]

(c) Now, suppose that the spatial dimensions are a 3-torus with side length L in each dimension. Assume that L is "large". By considering the periodicity constraint on the Hamiltonian, show that the torus constrains the degeneracy of each energy level (for each chirality) to $eBL^2/2\pi$.

(d) Suppose that

$$A^3(t=+\infty) - A^3(t=-\infty) = \frac{2\pi}{eL}$$

and that this change is performed adiabatically. Show that the number of right handed fermions decreases and the number of right handed fermions increases, and in particular that

$$\Delta N_{\mathrm{R}} - \Delta N_{\mathrm{L}} = -\frac{eBL^2}{\pi}.$$

(e) Show that this is precisely what is predicted by the QED anomaly formula for the quantum violation of the axial current.

[1] To find the energies, you should reduce the differential equation for the 2 component spinors to a second order differential equation, which should look familiar.