Soft Photons

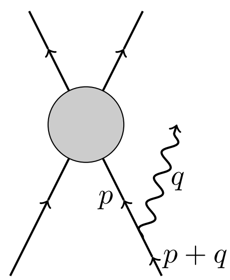
In this problem we explore the phenomenon of "soft photons," which are photons emitted in a process with energies much smaller than the characteristic energy scale of the problem. We will see that these processes are not negligible at any order in perturbation theory, *and* that with some effort we can actually compute them! They are divergent at low energies, but we can cancel these divergences.

In this problem, for simplicity, we will consider a QED theory consisting of a complex scalar field ϕ, with mass m and charge e, in $3+1$ dimensions:

$$\mathcal{L} = -\frac{1}{2}\mathrm{D}\overline{\phi}\cdot\mathrm{D}\phi - m^2\overline{\phi}\phi - \frac{1}{4}F^2$$

where $\mathrm{D}\phi = \partial\phi - \mathrm{i}eA\phi$ is the covariant derivative. With more work, you could show that similar things happen in QED with fermions, etc. The main reason it is so comparatively easy to work with high order corrections due to soft photons is due to the very simple way that we modify that the scattering amplitudes, as long as the energies of the photons are much smaller. For this part, assume that all the soft photons, and incoming particles, are real, and therefore their momenta are constrained to be on-shell.

(a) Consider the diagram



Show that the addition this soft photon adds to the scattering matrix amplitude is

$$\mathcal{M} \to \mathcal{M} \times \frac{ep^\mu}{p\cdot q - \mathrm{i}\epsilon}.$$

(b) Suppose that on the same incoming ϕ with momentum p, we emit n soft photons with momenta $q_1, \dots, q_n$. Show that after summing over perturbations, we find:

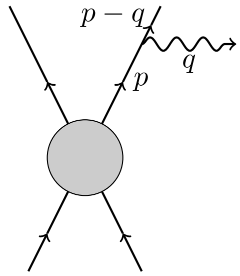
$$\mathcal{M} \to \mathcal{M} \times \prod_{i=1}^{n} \frac{ep^{\mu_i}}{p\cdot q_i - \mathrm{i}\epsilon},$$

i.e. prove the identity

$$\sum_{\pi\in\mathrm{S}_n}\prod_{i=1}^{n}\frac{ep^{\mu_{\pi(i)}}}{\sum_{j=1}^{i} p\cdot q_j - \mathrm{i}\epsilon} = \prod_{i=1}^{n}\frac{ep^{\mu_i}}{p\cdot q_i - \mathrm{i}\epsilon}.$$

I would recommend using mathematical induction to prove this.

(c) How would your answers to the above parts change if the soft photon was emitted after the process had occurred: e.g.,



(d) We will now compute the correction to a scattering amplitude, after taking into account these soft photons. Show that if we have a soft photon of momentum q, emitted from an unknown incoming scalar particle, the sum over all photon polarizations and possible emitting particles means:

$$|\mathcal{M}|^2 \to |\mathcal{M}^2| \frac{1}{(2\pi)^3 2q^0} \sum_{n,m} \frac{e^2 \eta_n \eta_m p_n \cdot p_m}{(p_n \cdot q)(p_m \cdot q)}$$

where the sum over n, m corresponds to all possible incoming/outgoing particles, and $\eta_n = 1$ if n is an incoming particle and -1 if it is an outgoing particle.

(e) In an experiment, we may not be able to realistically "see" all of the precise information about the photon, and so we are going to integrate over all possible soft photon emissions. Perform this integral:

$$\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2|\mathbf{q}|} \sum_{n,m} \frac{e^2 \eta_n \eta_m p_n p_m}{(p_n \cdot q)(p_m \cdot q)} = A \log \frac{\Lambda_1}{\Lambda_2}.$$

where

$$A = \sum_{n,m} \frac{e^2}{8\pi^2} \frac{\eta_n \eta_m}{\beta_{nm}} \log \frac{1 + \beta_{mn}}{1 - \beta_{mn}},$$

$$\beta_{mn} = \sqrt{1 - \frac{m^4}{(p_m \cdot p_n)^2}}.$$

Note that to do this integral, we need to restrict $\Lambda_1 < |\mathbf{q}| < \Lambda_2$. We will have more to say about these energy scales later. The integral over q is "elementary" but quite challenging – if you can't get it don't worry.

(f) Summing over the possibility of emitting any number of such soft photons from any leg, determine the change to $|\mathcal{M}|^2$. Note that as $\Lambda_1 \to 0$ it is very rapidly suppressed. We will soon see how to remove this.

We now proceed to show that the rapid decreasing of $|\mathcal{M}|^2$ of the previous part can be cancelled by considering virtual soft photons.

(g) Show that the addition of a virtual photon alters $\mathcal{M}$ by

$$\mathcal{M} \to \mathcal{M} \times -\frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{e^2 p_n \cdot p_m}{(q^2 - \mathrm{i}\epsilon)(p_n \cdot q - \mathrm{i}\eta_n \epsilon)(p_m \cdot q - \mathrm{i}\eta_m \epsilon)}$$

(h) We want to try and analyze the q integral above. Show that if $\eta_n \eta_m = -1$, then the integration reduces to the one from the previous section, but with a factor of $-1/2$. Conclude that the divergence of this factor in a scattering amplitude is canceled by the factor of the previous part.

(i) Show that if $\eta_m \eta_n = 1$, then the correction to $\mathcal{M}$ is only a phase, and thus unimportant for a scattering amplitude.

The analysis above is a bit oversimplified. In reality, there will be some energy scale Λ_{det} of the detector, and if the energy carried off by the real soft photons exceeds Λ_{det}, we will actually notice it. So we need to add a constraint to our analysis above.

(j) When summing up the possibility of emitting an arbitrary number of soft photons, include a step function restricting us to only look at processes where the total lost energy is below Λ_{det} via the following identity:

$$\Theta(a - x) = \int_{-\infty}^{\infty} \mathrm{d}z \frac{\sin(az)}{\pi z} \mathrm{e}^{\mathrm{i}xz}.$$

Show that now

$$|\mathcal{M}|^2 \to |\mathcal{M}|^2 \times F\left(\frac{\Lambda_2}{\Lambda_{\text{det}}}, A\right)$$

and determine the form of the modifying function F.

(k) Let us set $\Lambda_2 = \Lambda_{\text{det}}$. Then show that for $A \ll 1$,

$$F(1, A) \approx 1 - \frac{\pi^2}{12} A^2 + \cdots.$$

Therefore, the presence of soft photons slightly reduces the likelihood of the energetic process, although precisely how depends on the detection scheme. This should make sense, if only because losing more photons corresponds to a different process occurring, so we cannot count it in the scattering amplitude!