Motion with a Rapidly Oscillating Force

In this problem we will study how the motion of a particle is changed when it is placed in the presence of a rapidly oscillating force: i.e., consider a particle of mass m obeying Newton's Law:

$$m\ddot{x} = -V'(x) + f(x)\cos(\omega t)$$

where $V(x)$ is a potential energy function, and $f(x)$ is the position-dependent amplitude of the applied force. In particular, we are interested in the limit where ω corresponds to a "fast" time scale.

(a) As we have discussed, time scales are always relative. What time scale must $1/\omega$ be much smaller than for it to be considered fast?

Let us write

$$x(t) = X(t) + \xi(t)$$

where $X(t)$ is a slowly-varying function of time and $\xi(t)$ is a rapidly-varying function of time, which we will also take to be small.

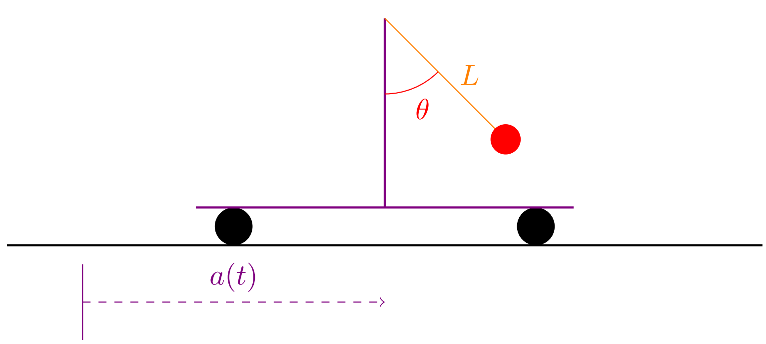
(b) By Taylor expanding V and f to lowest appropriate order in ξ and collecting the relevant terms, show that

$$\xi(t) \approx -\frac{f(X)}{m\omega^2}\cos(\omega t).$$

(c) Show that the motion for $X(t)$ can be described as the motion of a particle not under the action of an oscillating force, but in an effective potential

$$V_{\text{eff}}(x) \equiv V(x) + \frac{f(x)^2}{4m\omega^2}$$

This technique can be used to study some interesting problems to a good approximation. As an example, let us consider a pendulum of length L in a gravitational field of strength g which is placed on a moving cart. The position of the cart (on a 1D surface) is described by the function $a(t)$.



(d) Find the equation of motion for $\theta(t)$.[1]

[1] I would proceed by writing down the Lagrangian for $\theta(t)$ in the reference frame of the cart. Be careful!

(e) Assume now that

$$a(t) = A\cos(\omega t).$$

Following the logic we used earlier, find an effective potential for the motion of θ, and use it to determine the equilibria of the pendulum and their stabilities. Show that there is a critical value $\omega = \omega_{\mathrm{c}}$ at which the behavior of the pendulum will qualitatively change, and describe what happens for larger and smaller ω.

(f) Repeat the previous two parts, but now under the assumption that the cart's motion is vertical (parallel to g). Again, show that a critical value of ω at which the behavior qualitatively changes, and describe what happens for all ω.