Parametric Resonance

The equation of motion for a simple harmonic oscillator with a time-dependent frequency, $\omega(t)$, is

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}t} = M\boldsymbol{\eta}$$

where

$$\boldsymbol{\eta}(t) \equiv \left(\begin{array}{c} q(t) \\ p(t) \end{array} \right).$$

and

$$M(t) \equiv \left(\begin{array}{cc} 0 & m^{-1} \\ -m\omega(t)^2 & 0 \end{array} \right).$$

Suppose that $\omega(t)$ is periodic with period T: i.e., $\omega(t+T) = \omega(t)$ for all t. Define by A the linear operator such that

$$A\boldsymbol{\eta}(0) = \boldsymbol{\eta}(T).$$

Note that because the equation of motion is linear, A is in fact a good linear transformation.

(a) Explain why $\det(A) = 1$.

(b) Conclude that if $|\mathrm{tr}(A)| > 2$, the dynamics is not stable.

(c) Can we find a shortcut such as part (b) for determining the stability of higher dimensional oscillators with a periodic driven frequency "matrix"?

As a simple example, let us consider the system with

$$\omega(t) = \left\{ \begin{array}{ll} \omega_0 + \epsilon & 0 \le t < T/2 \\ \omega_0 - \epsilon & T/2 \le t < T \end{array} \right. , \quad \text{for } \epsilon \ll \omega_0.$$

This is a simple model for swinging on a swing set (something which I was not able to do as a kid).

(d) Find and sketch the regions in (ω_0, ϵ) space where the dynamics are stable or unstable. Comment on your results.

The phenomenon explored in this problem is called **parametric resonance**: altering the parameters of a system can, near certain resonances, cause a stable trajectory in phase space to become unstable.