Hadamard's Inequality

Information theory can be used to give a number of very beautiful proofs of interesting, and useful, matrix inequalities. In this problem, we will prove a particularly neat one which bounds the determinant of a symmetric matrix.

(a) Let's begin by generalizing our result that the Gaussian distribution maximizes differential entropy subject to constraint. Suppose we have a random variable $X \in \mathbb{R}^n$, with $\langle XX^{\mathrm{T}} \rangle = \Sigma$. Show that

$$\mathrm{h}(X) \leq \frac{n}{2} \log(2\pi \mathrm{e}) + \frac{1}{2} \log \det(\Sigma)$$

with equality $\iff$ X is multivariate Gaussian.

(b) Using this inequality, show that $\log \det(A)$ is a concave function of matrix $A \in \mathbb{R}^{n \times n}$. To do this, let $X_0, X_1 \in \mathbb{R}^n$ be multivariate Gaussian distributions, and $a \sim \mathrm{Bernoulli}(\lambda)$. Study $\mathrm{h}(X_a)$.

(c) Let $\Sigma \in \mathbb{R}^{n \times n}$ be symmetric. Using the previous result, prove **Hadamard's inequality**:

$$\det(\Sigma) \leq \prod_{i=1}^{n} \Sigma_{ii},$$

i.e., the determinant of a matrix is less than the product of its diagonal entries. When does equality hold?