Neural Spike Trains

A neuron transmits information in an animal by sending electrical signals down "cables" called axons. In this problem, we will consider a very simple model for the best possible information rate at which a neuron can possibly send signals in the animal. Roughly speaking, a neuron fires by creating a spike in voltage which can propagate down the channel – these electric spikes are generated by the cell pumping ions across the membrane separating the axon from its environment. The neuron has a characteristic time scale τ over which the ions may leak out through the membrane, and so roughly speaking there is an upper limit on how fast the neuron can send signals.

Let us suppose that the neuron is represented by a stochastic process $\{X_t\}$, with $X_t \in \mathbb{Z}_2$. We want to understand what is the maximal entropy process possible for the neuron, under the following constraint: if $X_t = 1$, then $X_{t+k} = 0$ for $1 \le k \le \tau$. This is a simple model for the neuron being unable to fire again until enough of the ions have depleted, which we assume takes a time τ. We are looking for the maximal rate of entropy transmission:

$$\alpha = \lim_{t\to\infty} \sup_{\mathrm{P}(X_1,\ldots,X_t)} \frac{\mathrm{H}(X_1,\ldots,X_t)}{t}.$$

(a) Justify the following statement: if$X_{t_k} = 1$ is the k^{th} time that we get a 1, and $X_{t_{k+1}} = 1$ is the $(k+1)^{\text{st}}$ 1, and we define $X_{t_{k+1}} - X_{t_k} = 1 + \tau + N_k$, and K is the total number of 1s in the first t time intervals, then

$$\mathrm{H}(X_1,\ldots,X_t) = \mathrm{H}(K) + \mathrm{H}(N_1,\ldots,N_k | K, N_1 + \cdots + N_K = t).$$

(b) Let $c_m = \mathrm{H}(N_1,\ldots,N_m | K = m, N_1 + \cdots + N_K = t)$. Assuming c_m is known, find the maximum possible entropy $\mathrm{H}(X_1,\ldots,X_t)$.

(c) What is the maximal possible c_m? (You should find a simple closed form answer, through combinatorics or another approach). You may use the following identity to simplify your answer:

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}.$$

(d) Assume $\tau \gg 1$. Show that, using asymptotic arguments,

$$\alpha \sim \frac{\log \tau}{\tau}.$$

(e) Comment on the physical/mathematical interpretation of the results. In particular, suppose each integer time step corresponds to an amount δ of real time. Let $\tau_0 = \delta\tau$ be fixed, and let $\tau \to \infty$. What happens to α? What does it suggest about the probability distribution on X_t (you don't need to actually find it)?