Martensites

Consider a solid structure undergoing a phase transition between 2 crystalline phases: for example, a transition where a cubic lattice stretches on one axis and compresses on another. At the interface between regions of the solid with these 2 phases, the large stresses could easily shatter the crystal. One way nature has found around this is to form martensites: intricate structures with large local but small global strains, which allow for the global structure to remain intact. This problem will study a very simple heuristic example of a martensitic free energy.

Consider the optimization problem of minimizing the free energy $F[u]$, for a 1D function $u(x)$ under the constraints that $u(0) = u(L) = 0$:

$$F[u] = \int_0^L \mathrm{d}x \left[\left((\partial_x u)^2 - 1 \right)^2 + au^2 + b \left(\partial_x^2 u \right)^2 \right].$$

Note that this free energy is not of a usual form: the solution $u = 0$ is *not* a minimum of F. Nonetheless, since F is always positive, we may think that there is some choice of u which is a minimum to F.

Let's begin by considering the case where $b = 0$.

(a) Find a sequence of trial functions $\{u_n(x)\}$ for which $F[u_{n+1}] < F[u_n]$, and for which

$$\lim_{n\to\infty} F[u_n] = 0.$$

(b) What does this suggest about the type of functions which minimize the free energy.[1]

Now, let us consider the case of generic b. Without finding the exact solution $u(x)$ which minimizes the free energy, we can understand it qualitatively. It should be clear from the first part of this problem that the solution will be a function $u(x)$ which folds to try and keep $\partial_x u \approx \pm 1$, although there will need to be crossover regimes between the two slopes.

(c) Show that the characteristic length/height of the emergent ripples, λ, is given by

$$\lambda \sim a^{-1/3} b^{1/6}.$$

[1] Martensite theory is an example of a place where the pathological functions of real analysis actually have physical interpretations!