Action of a Falling Object

In this problem, we will show that the principle of least action can, under naïve application, dramatically fail. Consider the simple case of an object of mass m, restricted to move on a vertical line with coordinate z. Furthermore, assume that this object falls only under the influence of Earth's gravitational field, with constant strength g pointing in the $-z$ direction.

(a) What is the Lagrangian L for this particle?

(b) What are Lagrange's equations?

(c) Use the initial conditions $z(0) = 0$ and $\dot{z}(0) = 0$. Plugging in to Lagrange's equations, what is the solution? Check that this is the physically appropriate solution.

Now, let us try and solve this problem directly from the principle of least action. Since that is hard, let us restrict to the ansatz $z(t) = At^2$, where A is some unknown constant. Certainly, $z(t)$ satisfies both our initial conditions, and thus is a good guess.

(d) Find the *minimum* of the action, assuming the trial functions $z(t)$ above. Show that you do not recover the physical solution. Furthermore, argue that the physical solution cannot even be an extremum of the action for our initial conditions.

(e) What is the resolution to this paradox?[1]

[1] Think back to the derivation of Lagrange's equations.