Domain Wall in a Ferromagnet

In the ordered phase of a ferromagnet, where the magnetization is a function of only a single spatial direction x, we can approximate the free energy as

$$F = \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\rho}{2} \left(\frac{\mathrm{d}M}{\mathrm{d}x} \right)^2 + \frac{\kappa}{4} \left(M^2 - M_0^2 \right)^2 \right].$$

The magnet will wish to adopt the state which minimizes the free energy F. In general, if the magnet chooses $M = \pm M_0$, then $F = 0$, which is its absolute minimum. However, sometimes this may not be possible. In this problem, we will consider what happens if the boundary conditions imposed are $M(-\infty) = -M_0$, and $M(\infty) = M_0$. Certainly, somewhere in space, we must have spatial gradients to flip the orientation of the magnet, even in the lowest possible state of free energy. The goal of this problem will be to compute the free energy cost of setting up such a **domain wall**, where the boundary conditions on the far ends of the wall are different.

(a) Write down Lagrange's equations for $M(x)$.

(b) Use Noether's Theorem to find a first order equation for $M(x)$. Use the boundary conditions, if necessary.

(c) Show that if we let $M(0) = 0$, the solution is

$$M(x) = M_0 \tanh \left(\sqrt{\frac{\kappa M_0^2}{2\rho}} x \right).$$

(d) Show that the free energy cost of the domain wall is

$$F = \sqrt{\frac{8\rho\kappa M_0^8}{9}}.$$