Seashells

In this problem, we will use variational calculus to determine the shape of a seashell. The theory of bent elastic solids suggests that given a curve parameterizing the central spine of the seashell, $x(\phi)$, $y(\phi)$ in 2D space, the elastic energy of such a curve is

$$E_{\text{elastic}}[x, y] = \int \mathrm{d}\phi \, \frac{1}{2}\kappa(\phi) \left[x^2 + y^2\right]$$

where ϕ is an angular parameter tracing out the curve, and κ is some parameter describing the "stiffness" of the shell at angle ϕ. There is also a "biological" contribution to the energy, which takes the form

$$E_{\text{bio}}[x, y] = -\int \mathrm{d}\phi \, \frac{1}{2}\kappa(\phi) \left[x\frac{\mathrm{d}y}{\mathrm{d}\phi} - y\frac{\mathrm{d}x}{\mathrm{d}\phi}\right].$$

Thus, the total energy functional is $E = E_{\text{elastic}} + E_{\text{bio}}$.

(a) To understand the meaning of the biological term, it will help to switch from rectangular coordinates x, y to polar coordinates r, θ, defined so that $x = r\cos\theta$ and $y = r\sin\theta$. Make this change of variables and find $E[r, \theta]$.

(b) Find the equations of motion, and their solution, given the initial conditions $\theta(0) = 0$ and $r(0) = r_0$. Comment on the purpose of the E_{bio} contribution to E.

(c) For a typical seashell, we might expect

$$\kappa(\phi) = \kappa_0 \mathrm{e}^{-2\alpha\phi}.$$

Sketch the resulting shape of the seashell.