The Geodesic Equation

On a curved space with coordinates $q^1 1, \ldots, q^n$, the Lagrangian for a free particle is given by

$$L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j.$$

Here, g_{ij} is a matrix called the **metric**, and it may depend on the coordinates q. The metric can be used to measure distances on a curved space: in particular, the distance between the points q^i and $q^i + \mathrm{d}q^i$ is given by

$$\mathrm{d}s^2 = g_{ij} \mathrm{d}q^i \mathrm{d}q^j.$$

The metric's inverse will also be relevant, and we denote it by g^{ij}:

$$g^{ij} g_{jk} = \delta^i_k.$$

This is not such a strange thing, in fact. You have likely seen the Lagrangian for motion constrained to live on a sphere of radius R. Labeling the coordinates as θ and ϕ, we may write the metric as

$$g = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}.$$

We see here that this metric does indeed depend on the coordinates.

In this problem, we will derive a very general result called the **geodesic equation**, which will describe how particles move on such a curved surface.

(a) Show that the equation of motion is

$$\ddot{q}^i + \Gamma^i{}_{jk} \dot{q}^j \dot{q}^k = 0$$

where $\Gamma^i{}_{jk}$ are the Christoffel coefficients, given by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial q_k} + \frac{\partial g_{mk}}{\partial q_j} - \frac{\partial g_{jk}}{\partial q_m} \right).$$

This is called the geodesic equation, and its solutions are called **geodesics**.

(b) In general, we may take the "dot product" of two vectors in the space time through the metric: i.e.

$$\mathbf{A} \cdot \mathbf{B} = g_{ij} A^i B^j.$$

Show that the speed, defined as $v \equiv \sqrt{\mathbf{v} \cdot \mathbf{v}}$, is constant on a geodesic.

(c) Show that the geodesic between two points is also the curve of minimum distance between those two points. No new math is required to do this.

The geodesic equation is one of the first ways to get introduced to the ideas of general relativity, where particles move on geodesics of the spacetime: in this case, time itself is treated as one of the coordinates, and a parameter such as "proper time" is used to parameterize the world line.