The Hall Effect in Noncommutative \mathbb{R}^2

Some models of M-theory and theories on branes in string theory predict that spacetime is noncommutative. In this problem, we will consider the noncommutative plane \mathbb{R}^2 . We can model this in the context of classical Hamiltonian mechanics by modifying the Poisson bracket to:

$$[f,g] = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} + \theta \epsilon_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}.$$

i = 1, 2 runs over the two spatial coordinates of \mathbb{R}^2 , and summation convention is implied over repeated indices, and $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = 1$ is the antisymmetric tensor in 2D. Here θ is a parameter controlling the strength of the non commutativity: when $\theta = 0$, the spacetime becomes commutative again and Hamiltonian mechanics returns to its normal form. The word noncommutative, for the context of this problem, should be thought of as simply stating that we have modified the Poisson bracket.

- (a) Find the Poisson brackets $[x_i, x_j]$, $[p_i, p_j]$ and $[x_i, p_j]$, and comment on the answer.
- (b) Using the definition

 $\dot{\xi} \equiv [\xi, H]$

where H is the Hamiltonian, determine Hamilton's equations for noncommutative \mathbb{R}^2 .

(c) Describe the dynamics of the free particle, with Hamiltonian given by

$$H = \frac{p_i p_i}{2m}$$

If there are difference from the commutative case, be sure to comment.

(d) Describe the dynamics of the simple harmonic oscillator, with Hamiltonian given by

$$H = \frac{p_i p_i}{2m} + \frac{m\omega^2 x_i x_i}{2}.$$

If there are difference from the commutative case, be sure to comment.

Now that we have some practice with Hamiltonian mechanics on noncommutative \mathbb{R}^2 , we are ready to focus on the most interesting application, which is to the motion of charged particles in magnetic fields. Let's turn to the minimal electromagnetic coupling of the Hamiltonian theory above, which can be written in terms of an action:

$$S = \int dt \left(\dot{x}_i \left(p_i + qA_i \right) - \frac{p_i p_i}{2m} - q\Phi + \frac{\theta}{2} \epsilon_{ij} p_i \dot{p}_j \right)$$

Here H_0 is the Hamiltonian of the particle in the absence of electromagnetic fields, and (Φ, A_i) is the covariant electromagnetic vector potential in 2 + 1 dimensions. We assume that other than the electromagnetic interactions, the particle is free for the remainder of the problem.

(e) What are the equations of motion? Express them in terms of the physical fields E_i and B.

- (f) Consider the case where the field B is a constant in space, and the field $E_i = 0$. What are the solutions of the equation of motion? Show that there is a critical value of B, B_c , for which there is an interesting switch in behavior.
- (g) Study the equations of motion more carefully at $B = B_c$. Assume that E_i is present and an arbitrary function of spacetime. Show that you can define twisted spatial coordinates

$$Q_i = x_i + \theta \epsilon_{ij} p_j$$

and that the equation of motions become

$$\dot{Q}_i = \epsilon_{ij} \frac{E_i}{B_c}$$

where the electric field is evaluated at the twisted coordinate. These are the equations of motion for a charged particle in the classical Hall effect in materials. Here they are arising naturally out of noncommutative geometry.

(h) Show that if we define the Poisson bracket, again working at $B = B_c$, to be

$$[f,g] \equiv -\theta \epsilon_{ij} \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial Q_j},$$

that the equations of motion of the previous part are given by Hamilton's equations, if we choose $H = q\Phi$.

(i) Compute $[Q_1, Q_2]$ using the Poisson bracket of the previous part. What is the relationship between Q_1 and Q_2 ?