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The Relativistic Kepler Problem

A relativistic particle with mass $m = 1$ has a Lagrangian given by, in units where $c = 1$:

$$L = -\sqrt{1 - \dot{r}^2 - r^2\dot{\theta}^2} + \frac{k}{r}.$$

(a) This may be a relativistic problem, but the tricks for solving central force problems will still apply. Discuss why both energy E and angular momentum l are still conserved, and find expressions for these quantities.

(b) Obtain the equivalent of Kepler's Third Law for relativistic particles, assuming that the orbit is circular.

(c) Show that there is a critical angular momentum l_c such that if $l < l_c$, the particle will spiral in towards $r = 0$.[1]

(d) Show that for a particle in an elliptical orbit, the ellipse will precess about the origin with some angular velocity Ω. Evaluate Ω in the non-relativistic limit.

[1] To do this, I would find a first order ODE for $r(\theta)$.