How Long Does Simulated Annealing Take?

The goal of this problem is to answer the question posed in the title. Let's begin with a preliminary fact.

(a) Prove the following mathematical proposition: given a sequence $a_n \in \mathbb{R}$, $n \in \mathbb{N}$, with $a_n \neq -1$, define

$$U = \prod_{n=1}^{\infty} (1 + a_n),$$

$$V = \sum_{n=1}^{\infty} |a_n|,$$

U converges to a nonzero value $\iff$ V is a convergent sum.

Now, let's turn back to simulated annealing. We want to understand whether simulated annealing can fail if we cool too fast (the intuition, as we discussed, is yes). Consider the following setup on state space $\Omega = \mathbb{Z}_4$: the energy function is $E(0) = E(2) = 2$, $E(1) = 1$ and $E(3) = 0$. For our Metropolis algorithm, we can transition from $x \to x \pm 1$ (with both being picked equally often).

We are concerned that this chain may get stuck at state 1, a local minimum. So let's consider a simulated annealing Markov chain X_t, with $X_0 = 1$.

(b) Suppose we had chosen

$$T_n = \frac{1}{a \log(n+1)}$$

for a some constant. Show that if $a > 1$ there is a nonzero probability that the chain stays forever at $X_t = 1$, but if $a \leq 1$ this event has probability 0. Lower bound these probabilities.

In general, if you want

$$\lim_{n \to \infty} \mathrm{P}(X_n \in \Omega_{\min}) = 1$$

where $\Omega_{\min} \subseteq \Omega$ corresponds to the elements in Ω with smallest energy, then you must have

$$T_n \geq \frac{c}{\log(1+n)}$$

for some constant c, depending on E and Ω.