Falling Raindrop

Consider a spherical droplet of water, with mass m and radius r, of density ρ_{w}, falling through a mist of density ρ_{m}. As the droplet falls through the mist, under gravitational field g, all the water contained in the mist is absorbed into the droplet, so the mass increases. Let $z(t)$ describe the distance towards the ground that the raindrop has traveled, and $v = \dot{z}$ the velocity of the raindrop. Assume that $z(0) = v(0) = 0$ – the raindrop initially starts at rest.

(a) Write down equations for $\dot{m}$, and $\dot{v}$, the rates of change of mass and the velocity of the raindrop.

(b) By relating m to r, show that the radius r is linearly related to z, and determine the linear coefficient of proportionality. Let r_0, the initial size of the raindrop, be the constant offset.

(c) Show that if $r_0 = 0$,

$$z(t) = \frac{1}{2}\frac{g}{7}t^2,$$

i.e., the raindrop falls under an effectively reduced gravitational field!

(d) The typical thickness of a rain cloud is about 2 km, and the relevant densities are $\rho_{\mathrm{w}} \approx 1000$ kg/m^3, and $\rho_{\mathrm{m}} \approx 10^{-3}$ kg/m^3. Estimate the size of a typical raindrop, and compare to the empirical result of ~ 1 mm. Use that $g \approx 10$ m/s^2.

(e) Show that, even when $r_0 \neq 0$, the velocity obeys the differential equation[1]

$$\ddot{v} = \frac{(g - 7\dot{v})(g - \dot{v})}{3v}$$

(f) Let $a = \dot{v}$. Use equations for $\dot{a}$ and $\dot{v}$ to qualitatively (but carefully) sketch the dynamics of raindrops in the (v, a) plane, for varying values of r_0. Be careful – the initial condition r_0 relates the values of a and v at $t = 0$. What is the early time dynamics? What is the late time dynamics?

In this model, we have neglected air resistance. We can approximate that air resistance leads to an extra force acting on the raindrop:

$$F_{\mathrm{air}} = -6\pi\eta r v.$$

(g) Using $\eta \approx 10^{-5}$ kg/m · s, determine whether or not air resistance will be important in the dynamics of a falling raindrop. Is neglecting air resistance legitimate for the long-time dynamics of raindrops?[2]

(h) Does the final mass of the raindrop, after it exits the cloud, change after accounting for air resistance?

(i) Estimate the terminal velocity, and kinetic energy, of a falling raindrop. Is the result reasonable? Be sure to take into account the result of the previous part, if needed.

[1] Take the derivative of an equation for $\dot{v}(z + k)$, where k is an appropriate constant.

[2] Be sure to consider the size of a cloud in your answer to this last question!